

压电声波器件分析理论与方法暑期讲习班
2012年8月6-9日，宁波大学

非线性电弹性力学

杨嘉实

Department of Mechanical and Materials Engineering
University of Nebraska-Lincoln
Lincoln, NE 68588-0526, USA

Outline

1. 连续介质运动学
2. 非线性电弹性理论
3. 线性压电理论
4. 结构振动的高频、准长波理论（线性）
5. 叠加在有限初始场（偏场）上的微小增量场（线性）

1. 连续介质运动学

Motion $y_i = y_i(\mathbf{X}, t)$

位移 $y_i = \delta_{iM}(X_M + u_M)$

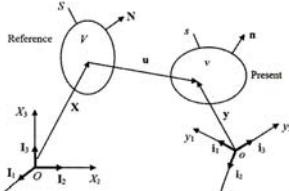
速度 $v_i = \frac{Dy_i}{Dt} = \dot{\delta}_{iM} = \frac{\partial y_i(\mathbf{X}, t)}{\partial t} \Big|_{\mathbf{X} \text{ fixed}}$

加速度 $\ddot{\delta}_{iM} = \frac{D^2y_i}{Dt^2} = \frac{\partial^2 y_i(\mathbf{X}, t)}{\partial t^2} \Big|_{\mathbf{X} \text{ fixed}}$

速度梯度, deformation rate, spin

$$\partial_i v_j = v_{j,i} = d_{ij} + \omega_{ij},$$

$$d_{ij} = \frac{1}{2}(v_{j,i} + v_{i,j}), \quad \omega_{ij} = \frac{1}{2}(v_{j,i} - v_{i,j}).$$



$$\mathbf{i}_k \cdot \mathbf{i}_l = \delta_{kl}, \quad \mathbf{I}_K \cdot \mathbf{I}_L = \delta_{KL}$$

$$\mathbf{i}_k \cdot \mathbf{I}_L = \delta_{kL}$$

$$\epsilon_{ijk} = \mathbf{i}_i \cdot (\mathbf{i}_j \times \mathbf{i}_k)$$

通常两个坐标系重合，
但仍认为是两个坐标系

连续介质运动学

变形梯度 $y_{k,K}$ Jacobian $J = \det(y_{k,K})$

$$\text{形变张量 } C_{KL} = y_{k,K} y_{k,L} = C_{LK}$$

$$\text{应变张量 } S_{KL} = (u_{K,L} + u_{L,K} + u_{M,K} u_{M,L}) / 2 = S_{LK}$$

$$S_{11} = u_{1,1} + (u_{1,1}u_{1,1} + u_{2,1}u_{2,1} + u_{3,1}u_{3,1}) / 2$$

$$S_{22} = u_{2,2} + (u_{1,2}u_{1,2} + u_{2,2}u_{2,2} + u_{3,2}u_{3,2}) / 2$$

$$S_{33} = u_{3,3} + (u_{1,3}u_{1,3} + u_{2,3}u_{2,3} + u_{3,3}u_{3,3}) / 2$$

$$S_{23} = (u_{2,3} + u_{3,2} + u_{1,2}u_{1,3} + u_{2,2}u_{2,3} + u_{3,2}u_{3,3}) / 2$$

$$S_{31} = (u_{3,1} + u_{1,3} + u_{1,3}u_{1,1} + u_{2,3}u_{2,1} + u_{3,3}u_{3,1}) / 2$$

$$S_{12} = (u_{1,2} + u_{2,1} + u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2}) / 2$$

应变率张量 $\dot{\delta}_{KL} = d_{ij} y_{i,K} y_{j,L}$

2. 非线性电弹性理论

- 发展简史
- 准静态近似
- 电介质的极化, 偶极近似
- 电场作用在带自由电荷、可极化、可变形电介质上的分布体力、体力偶及体功率
- 质量、线动量、角动量及能量守恒 (积分形式)
- 质量、线动量、角动量及能量守恒 (微分形式)
- Maxwell 应力张量
- 自由能
- 质量、线动量、角动量及能量守恒 (物质坐标形式)
- 本构关系
- 间断面条件
- 初边值问题
- 变分原理
- 总应力表述
- 各向同性材料
- 弱非线性的二次理论
- 三次理论
- 符号体系
- 非线性结构理论
- 非线性有限元



H. F. Tiersten in the winter of 1994-1995, RPI.

发展简史

- R. A. Toupin: The elastic dielectric. *J. Rational Mech. Anal.*, 5 (6), 849–915 (1956).
- H. F. Tiersten
- Eringen and Maugin: A. C. Eringen and G. A. Maugin, *Electrodynamics of Continua*. New York: Springer-Verlag, 1990.
- Nelson and Lax: D. F. Nelson, *Electric, Optic and Acoustic Interactions in Crystals*. New York: John Wiley and Sons, 1979.

准静态近似

在一特定波长（器件尺寸）下，声波频率远低于电磁波频率



低频振子（代表声波） 高频振子（代表电磁波）



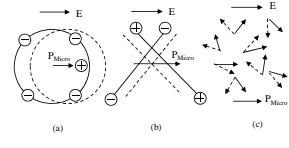
弱耦合系统具有两个自振频率：

- 在较低自振频率附近，高频振子行为近似为静止行为（在声波频率附近电磁场近似为静态）；
- 在较高自振频率附近，低频振子近似不动。

- D. F. Nelson and M. Lax, Linear elasticity and piezoelectricity in pyroelectrics, *Phys. Rev. B*, 13, 1785-1796, 1976.

电介质的极化，偶极近似

单位体积的极化矢量



$$\mathbf{P} = \lim_{\Delta v \rightarrow 0} \frac{\sum \mathbf{P}_{\text{Micro}}}{\Delta v}$$

不同的极化机理在宏观理论中不再区别

单位质量的极化矢量

$$\pi_i = P_i / \rho$$

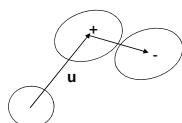
电位移矢量

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

偶极近似：高阶电矩（如四级矩）忽略

电场作用在带自由电荷、可极化、可变形电介质上的分布体力、体力偶及体功率

变形和极化后单位体积上的：



体力 $F_j^E = \rho_e E_j + P_i E_{j,i}$

体力偶 $C_i^E = \epsilon_{ijk} P_j E_k$

体功率 $w^E = \rho E_i \mathbf{A}_i$

微元体的移动、转动、变形与极化

($\rho_e = 0$)

Note: They are nonlinear. They depend on the unknown electric field and polarization.

Note: $\int_v -P_{i,i} E_j dv + \int_s P_i n_i E_j ds = \int_v P_i E_{j,i} dv$

质量、线动量、角动量及能量守恒（积分形式）

$$\int_s \mathbf{D} \cdot d\mathbf{a} = \int_v \rho_e dv,$$

$$\int_l \mathbf{E} \cdot d\mathbf{l} = 0,$$

$$\frac{D}{Dt} \int_v \rho dv = 0,$$

$$\frac{D}{Dt} \int_v \rho \mathbf{v} dv = \int_v (\rho \mathbf{f} + \mathbf{F}^E) dv + \int_s \mathbf{t} da,$$

$$\frac{D}{Dt} \int_v \mathbf{y} \times \rho \mathbf{v} dv = \int_v [\mathbf{y} \times (\rho \mathbf{f} + \mathbf{F}^E) + \mathbf{C}^E] dv + \int_s \mathbf{y} \times \mathbf{t} da,$$

$$\frac{D}{Dt} \int_v \rho \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} + e \right) dv = \int_v [(\rho \mathbf{f} + \mathbf{F}^E) \cdot \mathbf{v} + w^E] dv + \int_s \mathbf{t} \cdot \mathbf{v} da,$$

s, v, l: surface, volume, curve.

质量、线动量、角动量及能量守恒（微分形式）

$$D_{i,i} - \rho_e = 0$$

$$\nabla \times \mathbf{E} = \epsilon_{ijk} E_{k,j} \mathbf{i}_i = 0 \quad E_i = -\phi_i$$

Mass: $\rho \mathbf{A} + \rho v_{i,i} = 0$

Linear momentum: $\sigma_{ji,j} + \rho f_i + F_i^E = \rho \mathbf{A}$

Angular momentum: $\epsilon_{ijk} \sigma_{jk} + C_i^E = 0$

Energy: $\rho \mathbf{A} = \sigma_{ij} v_{j,i} + \rho E_i \mathbf{A}$

Maxwell 应力张量

$$\sigma_{ij}^E = F_j^E \quad \sigma_{ij}^E = D_i E_j - \frac{1}{2} \epsilon_0 E_k E_k \delta_{ij} = P_i E_j + \epsilon_0 (E_i E_j - \frac{1}{2} E_k E_k \delta_{ij})$$

由以上方式定义的 Maxwell 应力张量的不唯一性：Only its divergence is defined.

Maxwell 应力张量可以引入，也可以不引入。

Linear momentum: $(\sigma_{ij} + \sigma_{ij}^E)_j + \rho f_j = \rho \mathbf{A}$

Angular momentum: $\epsilon_{ijk} (\sigma_{jk} + P_j E_k) = \epsilon_{ijk} (\sigma_{jk} + \sigma_{jk}^E) = 0$

Total stress: $\tau_{ij} = \sigma_{ij} + \sigma_{ij}^E = \sigma_{ij} + P_i E_j + \epsilon_0 (E_i E_j - \frac{1}{2} E_k E_k \delta_{ij}) = \tau_{ji}$

Decomposition of the total stress:

$$\tau_{ij} = \sigma_{ij}^S + \sigma_{ij}^M$$

$$\sigma_{ij}^S = \sigma_{ij} + P_i E_j = \sigma_{ji}^S,$$

$$\sigma_{ij}^M = \epsilon_0 (E_i E_j - \frac{1}{2} E_k E_k \delta_{ij}) = \sigma_{ji}^M$$

自由能

Legendre transform: $\psi = e - E_i \pi_i$

Summary of equations:

$$\begin{aligned} D_{i,i} &= \rho_e \\ \rho \dot{\phi} + \rho v_{i,i} &= 0 \\ \tau_{ij,i} + \rho f_j &= \rho \dot{\phi} \\ \varepsilon_{ijk} \tau_{jk} &= 0 \\ \rho \dot{\psi} &= \sigma_{ij} v_{j,i} - P_i \dot{\phi} \end{aligned}$$

质量、线动量、角动量及能量守恒 (物质坐标形式)

Fields

$$\rho_E = \rho_e J$$

$$\mathcal{P}_K = J X_{K,k} P_k$$

$$\mathcal{E}_K = E_i y_{i,K} = -\dot{\phi}_i y_{i,K} = -\dot{\phi}_{,K}$$

$$\mathcal{D}_K = \varepsilon_0 J X_{K,i} D_i = \varepsilon_0 J C_{KL}^{-1} \mathcal{E}_L + \mathcal{P}_K$$

$$F_{Lj} = J X_{L,i} \sigma_{ij}^S \quad M_{Lj} = J X_{L,i} \sigma_{ij}^M$$

$$K_{Lj} = J X_{L,i} \tau_{ij} = F_{Lj} + M_{Lj}$$

$$T_{KL}^S = J X_{K,k} X_{L,l} \sigma_{kl}^S$$

Equations

$$\mathcal{D}_{K,K} = \rho_E$$

$$\rho_0 = \rho J$$

$$K_{Lk,L} + \rho_0 f_k = \rho_0 \dot{\psi}_K$$

$$\varepsilon_{kj} y_{i,L} K_{Lj} = 0$$

$$\rho_0 \dot{\psi} = T_{KL}^S \dot{\phi}_{KL} - \mathcal{P}_K \dot{\phi}_K$$

由微分形式的守恒律
经变量置换得到

一些未知量重组得到新未知量

质量、线动量、角动量及能量守恒 (物质坐标形式)

由积分形式的守恒律经变量置换得到。以线动量守恒为例：

$$\begin{aligned} \int_V \rho \dot{\phi} dV &= \int_V (\rho f_j + F_j^E) dV + \int_S t_j da = \int_V \rho f_j dV + \int_S \tau_j da_i \\ \int_V \rho \dot{\phi} dV &= \int_V \rho J \dot{\phi} dV = \int_V \rho J \dot{\phi} dV = \int_V \rho_0 \dot{\phi} dV \\ \int_V \rho f_j dV &= \int_V \rho f_j J dV = \int_V \rho J f_j dV = \int_V \rho_0 f_j dV \\ \int_S \tau_j da_i &= \int_S \tau_{ij} J X_{L,i} dA_L = \int_S \tau_{ij} J X_{L,i} N_L dA = \int_V (\tau_{ij} J X_{L,i})_{,L} dV \\ \int_V \rho_0 \dot{\phi} dV &= \int_V \rho_0 f_j dV + \int_V (\tau_{ij} J X_{L,i})_{,L} dV \\ K_{Lj,L} + \rho_0 f_j &= \rho_0 \dot{\phi} \end{aligned}$$

本构关系

$$\text{Assume} \quad \psi = \psi(S_{KL}, \mathcal{E}_K), \quad T_{KL}^S = T_{KL}^S(S_{KL}, \mathcal{E}_K), \quad \mathcal{P}_K = \mathcal{P}_K(S_{KL}, \mathcal{E}_K)$$

Energy equation

$$(T_{KL}^S - \rho_0 \frac{\partial \psi}{\partial S_{KL}}) \dot{\phi}_{KL} - (\mathcal{P}_K + \rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}) \dot{\phi}_K = 0.$$

$$T_{KL}^S = \rho_0 \frac{1}{2} \left(\frac{\partial \psi}{\partial S_{KL}} + \frac{\partial \psi}{\partial S_{LK}} \right) = \rho_0 \frac{\partial \psi}{\partial S_{KL}}, \quad \mathcal{P}_K = -\rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}$$

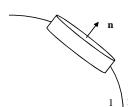
$$\text{Then} \quad K_{Lj} = y_{j,K} \rho_0 \frac{\partial \psi}{\partial S_{KL}} + y_{j,K} T_{KL}^M \quad \mathcal{D}_K = \varepsilon_0 J C_{KL}^{-1} \mathcal{E}_L - \rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}$$

$$\text{where} \quad T_{KL}^M = J X_{K,k} X_{L,l} \varepsilon_0 (E_k E_l - \frac{1}{2} E_m E_m \delta_{kl})$$

Note: The angular momentum equation is automatically satisfied.

间断面条件

Spatial form



$$\tau_y^{(2)} n_i - \tau_y^{(1)} n_i + \bar{t}_j = 0$$

$$D_i^{(2)} n_i - D_i^{(1)} n_i = \bar{\sigma}_e$$

物质间断面

Material form

$$K_{Lj}^{(2)} N_L - K_{Lj}^{(1)} N_L + \bar{T}_j = 0$$

$$\mathcal{D}_L^{(2)} N_L - \mathcal{D}_L^{(1)} N_L = \bar{\sigma}_E$$

初边值问题

方程

$$S_{KL} = (y_{i,K} y_{i,L} - \delta_{KL}) / 2$$

$$\mathcal{E}_K = -\dot{\phi}_K$$

$$\psi = \psi(S_{KL}, \mathcal{E}_K)$$

$$K_{Lj} = y_{j,K} \rho_0 \frac{\partial \psi}{\partial S_{KL}} + J X_{L,i} \varepsilon_0 (E_k E_l - \frac{1}{2} E_m E_m \delta_{kl})$$

$$S_y \cup S_T = S_\phi \cup S_D = S$$

$$S_y \cap S_T = S_\phi \cap S_D = 0$$

$$\mathcal{D}_K = \varepsilon_0 J C_{KL}^{-1} \mathcal{E}_L - \rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}$$

$$K_{Lj,L} + \rho_0 f_j = \rho_0 \dot{\phi}$$

4 equations for y and ϕ

$$\mathcal{D}_{K,K} = \rho_E$$

$$y_i = \bar{y}_i \quad \text{on} \quad S_y$$

$$\phi = \bar{\phi} \quad \text{on} \quad S_\phi$$

$$K_{Lk,L} N_L = \bar{T}_k \quad \text{on} \quad S_T$$

$$\mathcal{D}_K N_K = -\bar{\sigma}_E \quad \text{on} \quad S_D$$

边界条件

$$y_i = \bar{y}_i \quad \text{on} \quad S_y$$

$$\phi = \bar{\phi} \quad \text{on} \quad S_\phi$$

$$K_{Lk,L} N_L = \bar{T}_k \quad \text{on} \quad S_T$$

$$\mathcal{D}_K N_K = -\bar{\sigma}_E \quad \text{on} \quad S_D$$

非线性理论也不过是一些偏导数而已

变分原理

泛函

$$\begin{aligned}\Pi(\mathbf{y}, \boldsymbol{\phi}) = & \int_{t_0}^{t_1} dt \int_V \left[\frac{1}{2} \rho_0 \mathbf{y}^T \mathbf{y} - \rho_0 \psi(S_{KL}, E_K) \right. \\ & \left. + \pi(S_{KL}, W_K) + \rho_0 f_i y_i - \rho_E \phi \right] dV \\ & + \int_{t_0}^{t_1} dt \int_{S_p} \bar{T}_i y_i dS - \int_{t_0}^{t_1} dt \int_{S_p} \bar{\sigma}_E \phi dS,\end{aligned}$$

约束

$$\begin{aligned}\Pi(S_{KL}, E_K) = & \frac{1}{2} \epsilon_0 J E_k E_k \frac{1}{2} \epsilon_0 J C_{MN}^{-1} E_M E_N, \\ C_{KL} = & \delta_{KL} + 2S_{KL}, \\ S_{KL} = & (y_{i,k} y_{i,L} - \delta_{KL}) / 2, \\ \mathcal{E}_L = & -\phi_{,L},\end{aligned}$$

$$\begin{aligned}\delta\Pi = & \int_{t_0}^{t_1} dt \int_V \left[[K_{LL} + \rho_0 f_i - \rho_0 \mathbf{y}^T \mathbf{y}] \delta y_i + (\mathcal{D}_{LL} - \rho_E) \delta \phi \right] dV \\ & - \int_{t_0}^{t_1} dt \int_{S_p} (K_{LL} N_L - \bar{T}_i) \delta y_i dS \\ & - \int_{t_0}^{t_1} dt \int_{S_p} (\mathcal{D}_L N_L + \bar{\sigma}_E) \delta \phi dS.\end{aligned}$$

一阶变分

总应力表述

总自由能

$$\rho_0 \hat{\psi}(S_{KL}, E_K) = \rho_0 \psi(S_{KL}, E_K) - \pi(S_{KL}, E_K)$$

本构关系

$$K_{Lj} = y_{j,K} \rho_0 \frac{\partial \hat{\psi}}{\partial S_{KL}}, \quad \mathcal{D}_K = -\rho_0 \frac{\partial \hat{\psi}}{\partial E_K}$$

运动方程 (只含机械体力)

$$K_{Lj,L} + \rho_0 f_j = \rho_0 \mathbf{y}^T$$

与静电场方程

$$\mathcal{D}_{K,K} = \rho_E$$

变分原理

$$\begin{aligned}\Pi(\mathbf{y}, \boldsymbol{\phi}) = & \int_{t_0}^{t_1} dt \int_V \left[\frac{1}{2} \rho_0 \mathbf{y}^T \mathbf{y} - \rho_0 \hat{\psi}(S_{KL}, E_K) + \rho_0 f_i y_i - \rho_E \phi \right] dV \\ & + \int_{t_0}^{t_1} dt \int_{S_p} \bar{T}_i y_i dS - \int_{t_0}^{t_1} dt \int_{S_p} \bar{\sigma}_E \phi dS\end{aligned}$$

• 对应于不同的麦氏应力的选取, 有不同的总应力。

• 从总应力理论本身出发, 如果没有一个特定的麦氏应力表达式, 无法确定真实柯西应力及电场引起的分布体力。

各向同性材料

自由能

$$\rho_0 \psi = \Sigma(I_1, I_2, I_3, I_4, I_5, I_6)$$

不变量

$$\begin{aligned}I_1 = & \text{tr} S, \quad I_2 = \text{tr} S^2, \quad I_3 = \text{tr} S^3, \\ I_4 = & \mathbf{E} \cdot \mathbf{E}, \quad I_5 = \mathbf{E} \cdot \mathbf{S} \cdot \mathbf{E}, \quad I_6 = \mathbf{E} \cdot \mathbf{S}^2 \cdot \mathbf{E}.\end{aligned}$$

本构关系

$$\begin{aligned}\mathbf{T}^S = & \frac{\partial \Sigma}{\partial I_1} \mathbf{1} + 2 \frac{\partial \Sigma}{\partial I_2} S + 3 \frac{\partial \Sigma}{\partial I_3} S^2 + \frac{\partial \Sigma}{\partial I_5} \mathbf{E} \otimes \mathbf{E} + \frac{\partial \Sigma}{\partial I_6} [\mathbf{E} \otimes (\mathbf{S} \cdot \mathbf{E}) + (\mathbf{S} \cdot \mathbf{E}) \otimes \mathbf{E}] \\ \varphi = & -2 \frac{\partial \Sigma}{\partial I_4} \mathbf{E} - 2 \frac{\partial \Sigma}{\partial I_5} \mathbf{S} \cdot \mathbf{E} - 2 \frac{\partial \Sigma}{\partial I_6} S^2 \cdot \mathbf{E}\end{aligned}$$

弱非线性的二次理论

自由能

$$\rho_0 \psi(S_{KL}, E_K)$$

$$\begin{aligned}= & \frac{1}{2} \mathcal{C}_{ABCD} S_{AB} S_{CD} - e_{ABC} E_A S_{BC} - \frac{1}{2} \chi_{AB} E_A E_B \\ & + \frac{1}{6} \mathcal{C}_{ABCDEF} S_{AB} S_{CD} S_{EF} + \frac{1}{2} k_{ABCDE} E_A S_{BC} S_{DE} \\ & - \frac{1}{2} b_{ABCD} E_A E_B S_{CD} - \frac{1}{6} \chi_{ABC} E_A E_B E_C + \Lambda\end{aligned}$$

三阶材料常数
(非线性行为)

$$\mathcal{C}_{ABCDEF}, \quad k_{ABCDE}, \quad b_{ABCD}, \quad \chi_{ABC}$$

弱非线性的二次理论

应力

电位移

$$\mathcal{D}_K = \epsilon_0 J C_{KL}^{-1} E_L + \mathcal{P}_K$$

$$\begin{aligned}F_{Lj} = & F_{Lj} + M_{Lj} \\ F_{Lj} \equiv & \delta_{jkl} \left[\frac{1}{2} \mathcal{C}_{LMAB} u_{A,B} + e_{LMAB} \phi_{,A} + \frac{1}{2} \mathcal{C}_{LMAB} u_{K,A} u_{K,B} \right. \\ & \left. + \frac{1}{2} \mathcal{C}_{LMABCD} u_{A,B} u_{C,D} \right. \\ & \left. + e_{LMABCD} u_{M,A} \phi_{,A} + \frac{1}{2} d_{LMABCD} u_{B,C} u_{D,E} - b_{ALCD} u_{C,D} \phi_{,A} + \frac{1}{2} \chi_{ABL} \phi_{,A} \phi_{,B} \right] \\ M_{Lj} \equiv & \epsilon_0 \delta_{jkl} [\phi_{,L} \phi_{,M} - \frac{1}{2} \phi_{,L} \phi_{,K} \delta_{LM} \\ & - \phi_{,K} \phi_{,M} u_{K,L} - \phi_{,K} \phi_{,M} u_{L,K}], \\ \text{运动方程} \\ K_{Lj,L} + \rho_0 f_j = & \rho_0 \mathbf{y}^T\end{aligned}$$

静电场方程

三次理论

自由能

$$\rho_0 \psi(S_{KL}, E_K)$$

$$\begin{aligned}= & \frac{1}{2} \mathcal{C}_{ABCD} S_{AB} S_{CD} - e_{ABC} E_A S_{BC} - \frac{1}{2} \chi_{AB} E_A E_B \\ & + \frac{1}{6} \mathcal{C}_{ABCDEF} S_{AB} S_{CD} S_{EF} + \frac{1}{2} k_{ABCDE} E_A S_{BC} S_{DE} \\ & - \frac{1}{2} b_{ABCD} E_A E_B S_{CD} - \frac{1}{6} \chi_{ABC} E_A E_B E_C \\ & + \frac{1}{24} \mathcal{C}_{ABCDEFGH} S_{AB} S_{CD} S_{EF} S_{GH} + \frac{1}{6} k_{ABCDEFHG} E_A S_{BC} S_{DE} S_{FG} \\ & + \frac{1}{4} a_{ABCD} E_A E_B S_{CD} S_{EF} + \frac{1}{6} k_{ABCDEF} E_A E_B E_C S_{DE} \\ & - \frac{1}{24} \chi_{ABCD} E_A E_B E_C E_D + \Lambda\end{aligned}$$

四阶材料常数

$$\mathcal{C}_{ABCDEFGH}, \quad k_{ABCD}, \quad a_{ABCDEF}, \quad k_{ABCDEF}, \quad \chi_{ABCD}$$

Note: 几何非线性与物理非线性
H. F. Tiersten, Nonlinear electroelastic equations cubic in the small field variables, J. Acoust. Soc. Am., 57, 660-666, 1975.

三次理论

应力

位移

$$\begin{aligned}
 F_{ij} &= \delta_{ijkl} \left[\sum_{k,l,m} \epsilon_{ijkl} u_{i,k} + \epsilon_{ilm} \phi_{,l} - \frac{1}{2} \sum_{k,l,m} u_{i,k} u_{j,k} \right. \\
 &\quad + \epsilon_{ikm} u_{i,k} u_{j,k} + \frac{1}{2} \sum_{k,l,m} u_{i,k} u_{cd} \\
 &\quad + \epsilon_{ilm} u_{i,k} u_{j,k} - d_{ijkl} u_{i,k} \phi_{,l} - \frac{1}{2} b_{ijkl} \phi_{,l} \phi_{,k} \\
 &\quad + \frac{1}{2} \sum_{i,k,l,m} u_{i,k} u_{i,k} u_{j,k} + \frac{1}{2} \sum_{i,k,l,m} u_{i,k} u_{i,k} u_{cd} \\
 &\quad + \frac{1}{2} \sum_{i,k,l,m} u_{i,k} u_{i,k} u_{j,k} - \frac{1}{6} \sum_{i,k,l,m} u_{i,k} u_{cd} u_{j,k} \\
 &\quad - d_{ijkl} u_{i,k} u_{j,k} \phi_{,l} - \frac{1}{2} d_{ijkl} u_{i,k} u_{j,k} \phi_{,l} \\
 &\quad - \frac{1}{2} d_{ijkl} u_{i,k} u_{j,k} \phi_{,l} - \frac{1}{2} d_{ijkl} u_{i,k} u_{j,k} \phi_{,l} \\
 &\quad + \frac{1}{2} d_{ijkl} u_{i,k} u_{j,k} \phi_{,l} + \frac{1}{2} d_{ijkl} u_{i,k} u_{j,k} \phi_{,l} - \frac{1}{6} \sum_{i,k,l,m} u_{i,k} u_{j,k} \phi_{,l} \\
 &\quad + \frac{1}{2} \sum_{i,k,l,m} u_{i,k} u_{i,k} \phi_{,l} + \frac{1}{6} \sum_{i,k,l,m} u_{i,k} u_{j,k} \phi_{,l} \\
 M_{ij} &\equiv \epsilon_{ijkl} \left[\phi_{,l} \phi_{,M} - \frac{1}{2} \sum_{k,l,M} u_{i,k} u_{j,M} \right. \\
 &\quad - \phi_{,M} \phi_{,i} u_{i,k} - \phi_{,M} \phi_{,j} u_{j,k} - \frac{1}{2} \sum_{k,l,M} u_{i,k} u_{j,M} \\
 &\quad + \phi_{,i} \phi_{,j} u_{i,k} u_{j,k} + \frac{1}{2} \sum_{k,l,M} \phi_{,i} \phi_{,j} u_{i,M} - \frac{1}{2} \sum_{k,l,M} \phi_{,i} \phi_{,j} u_{j,M} \left. \right].
 \end{aligned}$$

符号体系

	Terminology	Notation	English Name	Chinese Name
Ref. coordinate	x_K	x_K	x_K	x_K
Present coordinate	y_i	y_i	y_i	y_i
Induction	$I = \det(y_{i,j})$	I	I	I
Ref. area density	α_0 (Symbol)	ρ^0	ρ^0	ρ^0
Present mass density	$\rho = \alpha_0 I$	ρ	ρ	ρ
Present charge density	ρ^e (From below)	ρ^e	ρ^e	ρ^e
Displacement	$u_j = y_j - x_{K,j}$	u_j	u_j	u_j
Velocity	$v_j = u_{j,t}$	v_j	v_j	v_j
Electric field potential grad.	$E_{i,j} = -\nabla_{x_{K,i}} u_{j,t}$	$E_{i,j}$	$E_{i,j}$	$E_{i,j}$
Mat. potential grad.	$H_{i,j} = E_{i,j} u_{j,t} - v_{i,t}$	$H_{i,j}$	$H_{i,j}$	$H_{i,j}$
Electric field	$P_i = \rho u_{i,t}$	P_i	P_i	P_i
Polarization	$P_i = \rho_e u_{i,t}$	P_i	P_i	P_i
Free surface	$\Gamma = \{y_i = 0\}$	Γ	Γ	Γ
Unit normal	$n_i = \pm \hat{n}_i$	n_i	n_i	n_i
Unit tangent	$t_i = n_i \times \hat{n}_i$	t_i	t_i	t_i
Electric displacement	$D_{i,j} = P_i + \chi A_{i,j}$	$D_{i,j}$	$D_{i,j}$	$D_{i,j}$
Electric field	$E_{i,j} = D_{i,j} / \epsilon_{i,j}$	$E_{i,j}$	$E_{i,j}$	$E_{i,j}$
Electric body force	$f_i = \rho g_i$	f_i	f_i	f_i
Electric body couple	$F_{i,j} = \rho_e g_{i,j}$	$F_{i,j}$	$F_{i,j}$	$F_{i,j}$
Charge per unit length	$q_i = \int_{\Gamma} P_i ds$	q_i	q_i	q_i
Two point force	$\mathbf{F}_{ij} = q_i t_i$	\mathbf{F}_{ij}	\mathbf{F}_{ij}	\mathbf{F}_{ij}
Unit force	$\mathbf{f}_{ij} = \mathbf{F}_{ij} / q_i$	\mathbf{f}_{ij}	\mathbf{f}_{ij}	\mathbf{f}_{ij}
Mixed electric current (vector)	$\mathbf{J}_{i,j}^0 = D_{i,j} E_{i,j} - \alpha_0 u_{i,t} \delta_{i,j}$	$\mathbf{J}_{i,j}^0$	$\mathbf{J}_{i,j}^0$	$\mathbf{J}_{i,j}^0$
Hydro. stress (spatial form)	$T_{i,j}^0 = \sigma_{i,j} + P_i t_i$	$T_{i,j}^0$	$T_{i,j}^0$	$T_{i,j}^0$
Transversal	$T_{i,j}^0 = \lambda K_{i,j} I^{-1}$ (Anisotropic)	$-$	$\lambda K_{i,j}$	$\lambda K_{i,j}$
Mat. form	$T_{i,j}^0 = \lambda K_{i,j} I^{-1}$ (Isotropic)	$-$	$\lambda K_{i,j}$	$\lambda K_{i,j}$
Hydro. Mixed stress (spatial form)	$T_{i,j}^0 = \lambda K_{i,j} I^{-1} + E_{i,j} F_{i,j} K_{i,j} / (2I)$	$T_{i,j}^0$	$\lambda K_{i,j}$	$\lambda K_{i,j}$
Transversal	$T_{i,j}^0 = \lambda K_{i,j} I^{-1} + E_{i,j} F_{i,j} K_{i,j} / (2I)$ (Anisotropic)	$-$	$\lambda K_{i,j}$	$\lambda K_{i,j}$
Mat. form	$T_{i,j}^0 = \lambda K_{i,j} I^{-1} + E_{i,j} F_{i,j} K_{i,j} / (2I)$ (Isotropic)	$-$	$\lambda K_{i,j}$	$\lambda K_{i,j}$
Unit stress	$\mathbf{T}_{i,j}^0 = \mathbf{E}_{i,j} \mathbf{F}_{i,j} K_{i,j} / (2I) + \mathbf{R}_{i,j} + \mathbf{M}_{i,j}$	$\mathbf{T}_{i,j}^0$	$\mathbf{T}_{i,j}^0$	$\mathbf{T}_{i,j}^0$
Free energy	$\psi = \psi_0 + \psi^{ext}$	ψ	ψ	ψ
Initial energy	$\psi_0 = \psi_0(\mathbf{u}, \mathbf{P}, \mathbf{E}, \mathbf{H})$	ψ_0	ψ_0	ψ_0
Final energy	$\psi = \psi(\mathbf{u}, \mathbf{P}, \mathbf{E}, \mathbf{H})$	ψ	ψ	ψ
Total free energy	$\psi = \psi_0 + \psi^{ext}$	ψ	ψ	ψ

J. S. Yang, On the notations of the nonlinear theory of electroelasticity, IEEE Trans. Ultrason., Ferroelect., Freq. Contr., 54, 2702-2704, 2007.

非线性结构理论

• Von Karman 理论: 板的大挠度弯曲

• 板的非线性大剪切变形理论, Poynting 效应, 非线性模态耦合:

非线性厚度剪切 (thickness shear)

J. S. Yang, X. M. Yang, J. A. Turner, J. A. Kosinski and R. A. Pastore, Jr., Two-dimensional equations for electroelastic plates with relatively large shear deformations, IEEE Trans. Ultrason., Ferroelect., Freq. Contr., 50, 765-772, 2003.

非线性面内剪切 (face shear, 平面应力)

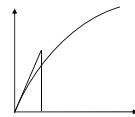
J.S. Yang, Two-dimensional equations for electroelastic plates with relatively large in-plane shear deformation and nonlinear mode coupling in resonant piezoelectric devices, Acta Mechanica, 196, 103-111, 2008.

• 较一般的非线性板理论: Ji Wang, et al.

• 杆拉伸的非线性理论: Tiersten and Smythe

3. 线性压电理论

非线性理论在零点附近的线性化



线性理论参考文献

- [1] A. H. Meitzler, D. Berlincourt, F. S. Welsh, III, H. F. Tiersten, G. A. Coquin and A. W. Warner, *IEEE Standard on Piezoelectricity*, IEEE, New York, 1988.
- [2] H. F. Tiersten, *Linear Piezoelectric Plate Vibrations*, Plenum, New York, 1969.
- [3] J. S. Yang, *An Introduction to the Theory of Piezoelectricity*. Springer, New York, 2005.

非线性有限元



线性化

$$\|u_{i,K}\| < 1 \quad \phi_{,K} \text{ is infinitesimal}$$

$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}$$

$$\rho_0 \equiv \rho(1+u_{k,k})$$

$$\rho_0 \dot{\psi}(S_{kl}, E_k) \rightarrow \frac{1}{2} c_{ijkl}^E S_{ij} S_{kl} - e_{ijk} E_i S_{jk} - \frac{1}{2} \epsilon_y^S E_i E_j = H(S_{kl}, E_k)$$

$$T_{ij} = \frac{\partial H}{\partial S_{ij}} = c_{ijkl}^E S_{kl} - e_{ijk} E_k, \quad D_i = \frac{\partial H}{\partial E_i} = e_{ikl} S_{kl} + c_{ik}^S E_k$$

$$T_{i,j,i} + \rho f_i = \rho \mathbf{f}_i$$

$$D_{i,i} = \rho_e$$

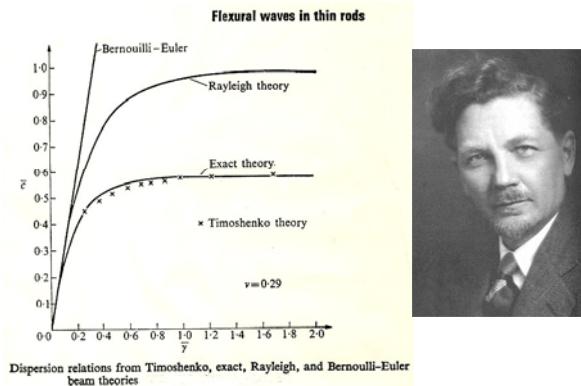
$$D_{i,i} = \rho_e u_{k,i} - \epsilon_y^S \phi_{,i}$$

$$D_{i,i} = \rho_e$$

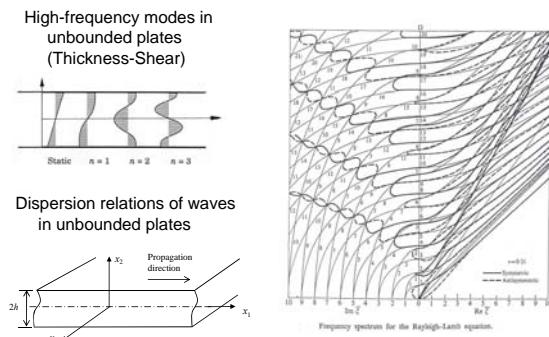
4. 结构振动的高频、准长波理论（线性）

- 梁弯曲的Timoshenko理论
- 板弯曲的高频理论
- 板高频厚度剪切与厚度扭转振动的 Stevens-Tiersten方程
- 板拉伸的高频理论
- 杆拉伸的高频理论
- 杆扭转的高频理论

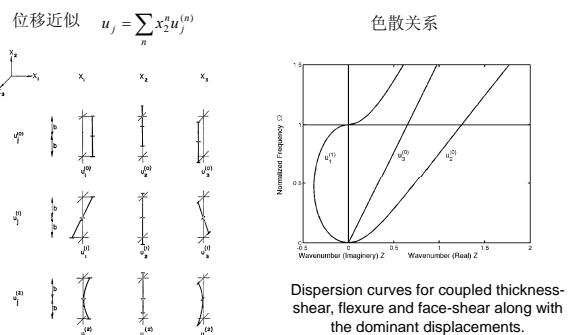
梁弯曲的Timoshenko理论



板的低频与高频振动



板弯曲的高频理论: Mindlin-Reissner



Mindlin RD. An Introduction to the Mathematical Theory of Vibrations of Elastic Plates. Singapore: World Scientific, 2006 (Yang JS ed.).

Mindlin 学派

R. D. Mindlin (1906-1987, Columbia University, NAS, NAE) Advisor?



- Presidential Medal for Merit in 1946. For Proximity fuse in WWII.
- Nat. Medal of Sci., 1979. For fundamental contributions to applied mechanics, including theory and applications in photoelasticity, package cushioning, piezoelectric oscillators, and ultrahigh frequency vibrations."



Mindlin 学派: Students

- Bleustein, Jeffrey Ph.D. 1965 Yale, now Harley & Davidson
- Bogdanoff, John Ph.D. 1949 Cornell
- Brady, Kevin DES 1964 Bell Labs
- Cheng, David Ph.D. 1944 CUNY
- Deresiewicz, Herbert Ph.D. 1952 Columbia
- Drucker, Daniel Ph.D. 1938 UIUC
- Duffy, Jacques Ph.D. 1957 Brown
- Eshel, Nachman Ph.D. 1965 CUNY
- Forray, Martin Ph.D. 1955 C. W. Post
- Fox, Edward Ph.D. 1958 RPI
- Gazis, Denas Ph.D. 1957 IBM
- Gong, Chung DES 1970 Ebsco
- Goodman, Lawrence Ph.D. 1949 Univ. of Minnesota
- Haines, Daniel Ph.D. 1968 Univ. of South Carolina
- Hopmann, W. H. Ph.D. 1947 Univ. of South Carolina
- Huang, Y. T. Ph.D. 1961 Huang & Associate
- Kane, Thomas R. Ph.D. 1953 Stanford
- Kaul, Raj Ph.D. 1963 SUNY Buffalo
- Lee, Peter C. Y. DES 1963 Princeton
- Lubkin, J. L. Ph.D. 1964 Michigan State
- Lubowe, Anthony DES 1961 Bell Labs
- McNiven, Hugh Ph.D. 1958 Berkeley
- McCoy, John Catholic Univ.
- Mediek, Mathew Ph.D. 1957 Michigan State
- Newman, E. G. IBM
- Pao, Yih-Hsing Ph.D. 1959 Cornell, now Zhejiang University
- Robinstion, Kenneth ITEK Corp.
- Rongved, Leif Ph.D. 1953 Bell Labs
- Schwartz, Jeremy DES 1969 Naval Weapons Lab
- Tasi, James Ph.D. 1961 SUNY Stony Brook
- Tiersten, Harry Ph.D. 1961 RPI

Mindlin 学派: Books by or about Mindlin

•Mindlin RD. *An Introduction to the Mathematical Theory of Vibrations of Elastic Plates*. Singapore: World Scientific, 2006 (Yang JS ed.).



•Mindlin RD and Lee PCY: To appear.

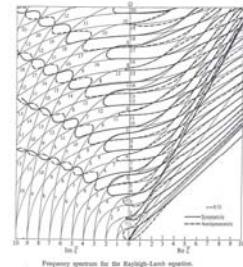
•Herrmann G (ed). *R.D. Mindlin and Applied Mechanics*, Pergamon Press, 1974.

•Deresiewicz H, Bieniek MP and DiMaggio FL (ed). *The Collected Papers of Raymond D. Mindlin*. Springer-Verlag, 1989.

板高频厚度剪切与厚度扭转振动的 Stevens-Tiersten 方程

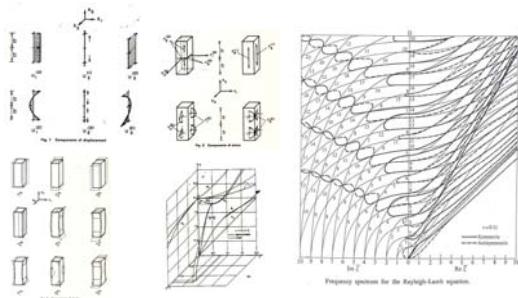
$$u_1(x_1, x_2, x_3) \equiv \sum_{n=1,3,5,\Delta}^{\infty} u_n^n(x_1, x_2) \sin \frac{n\pi}{2h} x_3$$

$$M_s \frac{\partial^2 u_n^n}{\partial x_1^2} + Q_s \frac{\partial^2 u_n^n}{\partial x_1 \partial x_3} + P_s \frac{\partial^2 u_n^n}{\partial x_3^2} - \frac{n^2 \pi^2 C^{(1)}}{4h^2} u_n^n - \rho \frac{d^2 u_n^n}{dt^2} = 0$$



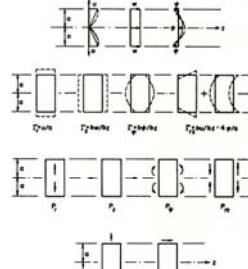
- D. S. Stevens and H. F. Tiersten, An analysis of doubly rotated quartz resonators utilizing essentially thickness modes with transverse variation, *J Acoust Soc Am*, **79**, 1811-1826, 1986.
- E. P. EerNisse, Analysis of thickness modes of contoured, doubly rotated, quartz resonators, *IEEE Trans. Ultrason., Ferroelectr., Freq. Contr.*, **48**, 1351-1361, 2001.

板拉伸的高频理论 (Mindlin-Medick)



- Kane, T. R., Mindlin, R. D.: High-frequency extensional vibrations of plates. *ASME J. Appl. Mech.* 23, 277-283 (1956). 不正确: 不能描述复色散关系。
- Mindlin, R. D., Medick, M. A.: Extensional vibrations of elastic plates. *ASME J. Appl. Mech.* 26, 561-569 (1959).

杆拉伸的高频理论 (Mindlin-McNiven)



色散关系

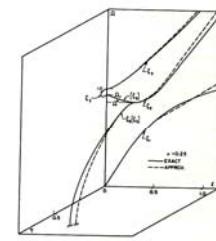
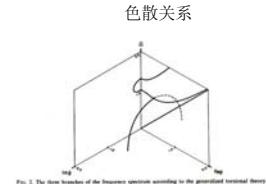
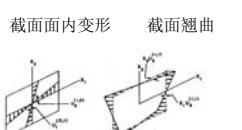


Fig. 1 Displacements, strains, stresses, and surface tractions; second-order approximation

- R. D. Mindlin and G. Herrmann, A one-dimensional theory of compressional waves, in an elastic rod. Proceedings of the First U.S. National Congress of Applied Mechanics, 1952, 187-191. 不正确: 不能描述复色散关系。
- R. D. Mindlin and H. D. McNiven, "Axially symmetric waves in elastic rods," *ASME Journal of Applied Mechanics*, 27, 145-151, 1960.

杆扭转的高频理论: Bleustein-Stanley

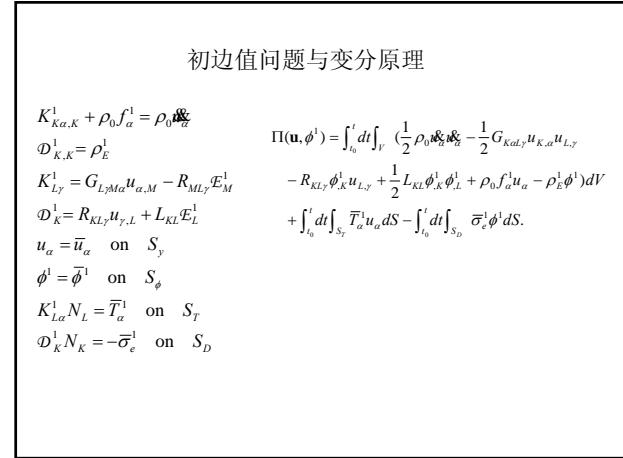
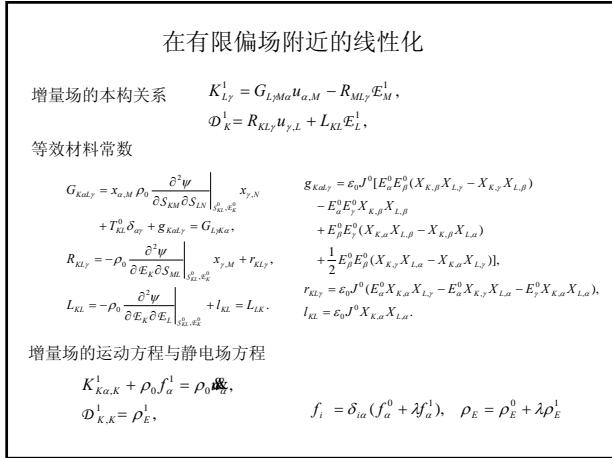
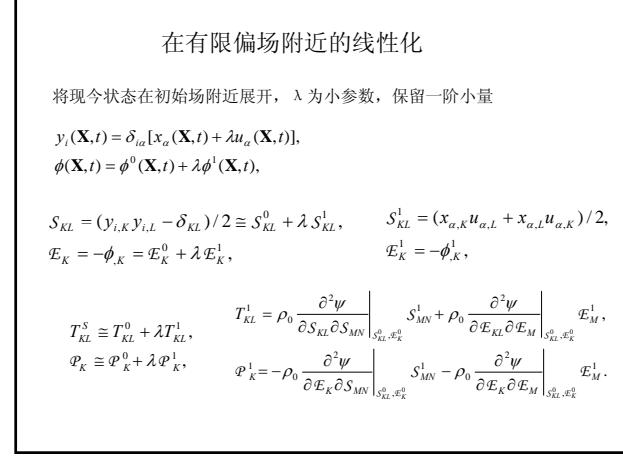
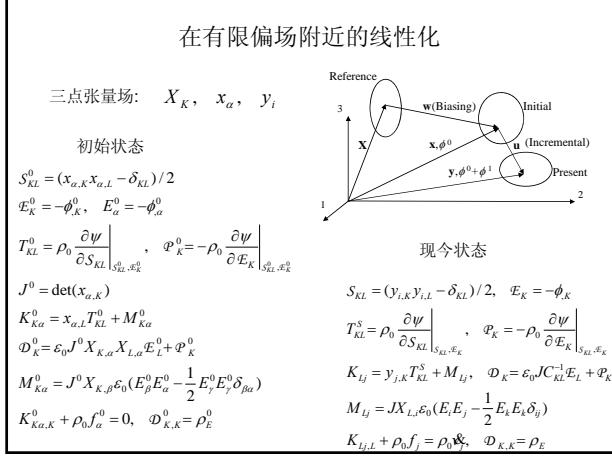
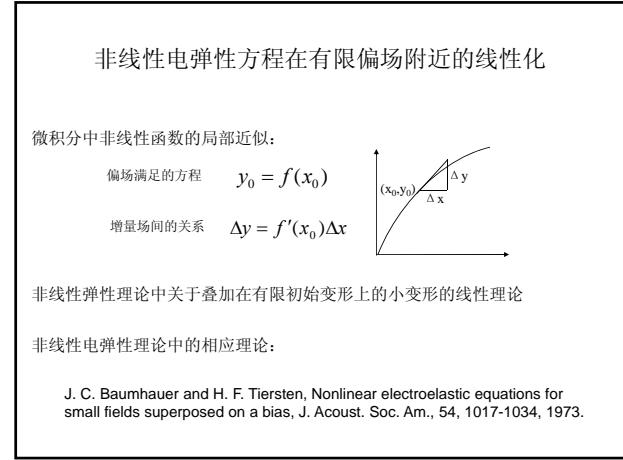
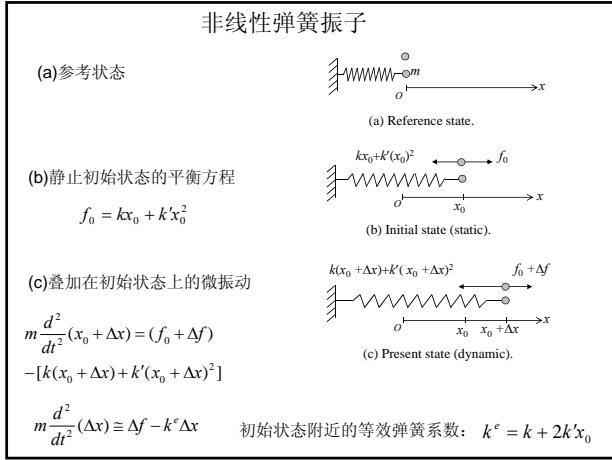


Jeffrey L. Bleustein, Richard M. Stanley, A dynamical theory of torsion, *International Journal of Solids and Structures*, Volume 6, Issue 5, May 1970, Pages 569-586



5. 叠加在有限初始场 (偏场) 上的微小增量场 (线性)

- 非线性弹簧振子
- 非线性电弹性方程在有限偏场附近的线性化
- 初边值问题与变分原理
- 小偏场的情况
- 与初应力理论的关系
- 各向同性材料在偏场下的等效压电系数
- 晶体振荡器频率稳定性
- 关于自由振动频率的一阶摄动积分
- 二阶摄动积分
- 有简并时的摄动
- 梁、板、壳理论
- 非线性材料常数的测定



小偏场的情况

$$\rho_0 \psi(E_{KL}, E_K) = \frac{1}{2} C_{ABCD} S_{AB} S_{CD} - e_{ABC} E_A S_{BC} - \frac{1}{2} \chi_{AB} E_A E_B + \frac{1}{6} C_{ABCDE} S_{AB} S_{CD} S_{EF} + \frac{1}{2} k_{ABCD} E_A S_{BC} S_{DE} - \frac{1}{2} b_{ABCD} E_A E_B S_{CD} - \frac{1}{6} \chi_{ABC} E_A E_B E_C$$

小偏场 $x_\alpha = \delta_{\alpha K} X_K + w_\alpha \quad S_{AB}^0 \cong (w_{A,B} + w_{B,A}) / 2 \quad E_K^0 = -\phi_K^0$

$$T_{KL}^0 = c_{KLMN} S_{MN}^0 - e_{MLK} E_M^0$$

等效材料常数

$$G_{Kal\gamma} = c_{Kal\gamma} + \hat{c}_{Kal\gamma} + c_{Kal\gamma AB} S_{AB}^0 + k_{AKal\gamma} E_A^0$$

$$\hat{e}_{KL\gamma} = e_{KLM} w_{\gamma M} - k_{KL\gamma AB} S_{AB}^0 + b_{AKL\gamma} E_A^0$$

$$R_{KL} = \epsilon_{KL} + \hat{\epsilon}_{KL}$$

$$L_{KL} = \epsilon_{KL} + \hat{\epsilon}_{KL} + e_0 (E_K^0 \delta_{L\gamma} - E_L^0 \delta_{K\gamma} - E_M^0 \delta_{M\gamma} \delta_{KL})$$

$$\hat{\epsilon}_{KL} = b_{KLAB} S_{AB}^0 + \chi_{KLA} E_A^0 + e_0 (\chi_{MM} \delta_{KL} - 2 S_{KL}^0)$$

Note: The 3rd-order material constants are needed for a complete description of the 1st-order effects of the biasing fields.

与初应力理论的关系

• 初应力理论以右图中初始状态作为参考状态。该状态随初始场变化，作为参考状态在应用中常常不便。

• 在偏场理论中令 $\mathbf{x}=\mathbf{X}$ 可得到初应力理论：

$$G_{Kal\gamma} = c_{Kal\gamma} + \hat{c}_{Kal\gamma}$$

$$R_{KL\gamma} = e_{KL\gamma} + \hat{e}_{KL\gamma}$$

$$L_{KL} = \epsilon_{KL} + \hat{\epsilon}_{KL}$$

$$\hat{c}_{Kal\gamma} = T_{KL}^0 \delta_{\alpha\gamma} + k_{AKal\gamma} E_A^0$$

$$\hat{e}_{KL\gamma} = b_{AKL\gamma} E_A^0 + e_0 (E_K^0 \delta_{L\gamma} - E_L^0 \delta_{K\gamma} - E_M^0 \delta_{M\gamma} \delta_{KL})$$

$$\hat{\epsilon}_{KL} = \chi_{KLA} E_A^0$$

各向同性材料在偏场下的等效压电系数

自由能 $\rho_0 \psi = c_1 I_1^2 + c_2 I_2 - \varepsilon_0 \chi I_4 / 2 + a_1 I_1^3 + a_2 I_1 I_2 + a_3 I_3 + b_1 I_1 I_4 + b_2 I_5 + \dots$

不变量 $I_1 = \text{tr} S, \quad I_2 = \text{tr} S^2, \quad I_3 = \text{tr} S^3, \quad I_4 = E \cdot S \cdot E, \quad I_5 = E \cdot S \cdot E, \quad I_6 = E \cdot S^2 \cdot E$

初始变形 $x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3$

初始电场 \mathbf{W}^0

等效压电系数

$R_{211} = -2\lambda_1 W_1^0 b_1 + \lambda_1^{-1} \lambda_2^{-2} \varepsilon_0 J_0 W_1^0,$	$R_{311} = -2\lambda_1 W_3^0 b_1 + \lambda_1^{-1} \lambda_3^{-2} \varepsilon_0 J_0 W_3^0,$
$R_{212} = -\lambda_2 W_1^0 b_2 - \lambda_1^{-2} \lambda_2^{-1} \varepsilon_0 J_0 W_1^0,$	$R_{312} = 0,$
$R_{213} = 0,$	$R_{313} = -\lambda_2 W_1^0 b_2 - \lambda_1^{-2} \lambda_3^{-1} \varepsilon_0 J_0 W_1^0,$
$R_{221} = -\lambda_1 W_1^0 b_2 - \lambda_1^{-1} \lambda_2^{-2} \varepsilon_0 J_0 W_1^0,$	$R_{321} = 0,$
$R_{222} = -2\lambda_1 W_1^0 (b_1 + b_2) - \lambda_1^{-3} \varepsilon_0 J_0 W_1^0,$	$R_{322} = -2\lambda_1 W_3^0 b_1 + \lambda_1^{-1} \lambda_3^{-2} \varepsilon_0 J_0 W_3^0,$
$R_{223} = -\lambda_3 W_3^0 b_2 - \lambda_2^{-2} \lambda_3^{-1} \varepsilon_0 J_0 W_3^0,$	$R_{323} = -\lambda_2 W_2^0 b_2 - \lambda_2^{-2} \lambda_3^{-1} \varepsilon_0 J_0 W_2^0,$
$R_{231} = 0,$	$R_{331} = -\lambda_1 W_2^0 b_2 - \lambda_1^{-1} \lambda_3^{-1} \varepsilon_0 J_0 W_2^0,$
$R_{232} = -\lambda_2 W_3^0 b_2 - \lambda_2^{-1} \lambda_3^{-2} \varepsilon_0 J_0 W_3^0,$	$R_{332} = -\lambda_1 W_2^0 b_2 - \lambda_2^{-1} \lambda_3^{-1} \varepsilon_0 J_0 W_2^0,$
$R_{233} = -2\lambda_1 W_2^0 b_2 + \lambda_2^{-2} \lambda_3^{-1} \varepsilon_0 J_0 W_2^0,$	$R_{333} = -2\lambda_2 W_3^0 (b_1 + b_2) - \lambda_3^{-3} \varepsilon_0 J_0 W_3^0,$

晶体谐振器 (resonator) 与振荡器 (Oscillator)

提供频率标准，用于计时、信号处理、传感等。

晶体振荡器频率稳定性

- Temperature**
 - Static frequency vs. temperature
 - Dynamic frequency vs. temperature (warmup, thermal shock)
 - Thermal history ("hysteresis," "retrace")
- Acceleration**
 - Gravity (2g tipover)
 - Vibration
 - Acoustic noise
 - Shock
- Time**
 - Short term (noise)
 - Intermediate term (e.g., due to oven fluctuations)
 - Long term (aging)
- Ionizing radiation**
 - Steady state
 - Pulsed
 - Photons (X-rays, γ -rays)
 - Particles (neutrons, protons, electrons)
- Other**
 - Power supply voltage
 - Atmospheric pressure (altitude)
 - Humidity
 - Load impedance
 - Magnetic field

SOURCE: Quartz Crystal Resonators and Oscillators for Frequency Control and Timing Applications, A Tutorial by John R. Vig. Follow links to tutorials at <http://www.kicsoft.com>

关于自由振动频率的一阶摄动积分

有偏场时的自由振动特征值问题

$$-(c_{LMa} + \hat{c}_{LMa}) u_{a,M} + (e_{ML\gamma} + \hat{e}_{ML\gamma}) \phi_{,M} \Big|_L = \rho_0 \lambda u_\gamma, \quad \text{in } V,$$

$$-(e_{KL\gamma} + \hat{e}_{KL\gamma}) u_{\gamma,L} + (e_{KL} + \hat{e}_{KL}) \phi_{,L} \Big|_K = 0, \quad \text{in } V,$$

$$u_a = 0, \quad \text{on } S_a,$$

$$K_{L,T}^1 N_L = [c_{LMa} + \hat{c}_{LMa}] u_{a,M} + (e_{ML\gamma} + \hat{e}_{ML\gamma}) \phi_{,M}^1 \Big|_V = 0, \quad \text{on } S_T,$$

$$\phi^1 = 0, \quad \text{on } S_\phi,$$

$$\mathcal{D}_K^1 N_K = [(e_{KL\gamma} + \hat{e}_{KL\gamma}) u_{\gamma,L} - (e_{KL} + \hat{e}_{KL}) \phi_{,L}^1] N_K = 0, \quad \text{on } S_D,$$

固有频率一阶摄动积分

$$\frac{\omega - \omega^{(0)}}{\omega^{(0)}} \cong \frac{1}{2(\omega^{(0)})^2} \int_V \frac{1}{\rho_0 u_a^{(0)} u_a^{(0)}} dV$$

$$\times \int_V (\varepsilon \hat{c}_{LMa} u_{a,M}^{(0)} u_{\gamma,L}^{(0)} + 2\varepsilon \hat{e}_{ML\gamma} \phi_{,M}^{(0)} u_{\gamma,L}^{(0)} - \varepsilon \hat{e}_{KL} \phi_{,L}^{(0)} \phi_K^{(0)}) dV.$$

Limitations:

- ω is simple (no degeneracy)
- ω is isolated
- w (bias) is infinitesimal
- w is time-independent

H. F. Tiersten, Perturbation theory for linear electroelastic equations for small fields superposed on a bias, J. Acoust. Am., 64, 832-837, 1978.

Application of the 1st-order Perturbation



Patriot (By Raytheon)



SAW Oscillator

Current technology: relative frequency shift = $10^{-10}/g$ Goal of the US Army: from the current $10^{-10}/g$ technology to $10^{-12}/g$ in the near future

二阶摄动积分

Eigenvalue problem: $\lambda = \omega^2$

$$-\left[(c_{L,M\alpha} + \varepsilon \bar{c}_{L,M\alpha} + \varepsilon^2 \tilde{c}_{L,M\alpha}) u_{\alpha,M} + (e_{M\gamma} + \varepsilon \bar{e}_{M\gamma} + \varepsilon^2 \tilde{e}_{M\gamma}) \phi_M^1 \right]_L = \rho_0 \lambda u_\gamma \\ -\left[(e_{K,L\gamma} + \varepsilon \bar{e}_{K,L\gamma} + \varepsilon^2 \tilde{e}_{K,L\gamma}) u_{\gamma,L} + (\varepsilon_{KL} + \varepsilon \bar{\varepsilon}_{KL} + \varepsilon^2 \tilde{\varepsilon}_{KL}) \phi_L^1 \right]_K = 0$$

Abstract formulation: $(\mathbf{A} + \varepsilon \bar{\mathbf{A}} + \varepsilon^2 \tilde{\mathbf{A}}) \mathbf{U} = \lambda \mathbf{B} \mathbf{U}$

$$\text{Perturbation expansions: } \lambda \equiv \lambda^{(0)} + \varepsilon \lambda^{(1)} + \varepsilon^2 \lambda^{(2)} \\ \mathbf{U} = \mathbf{U}^{(0)} + \varepsilon \mathbf{U}^{(1)} + \varepsilon^2 \mathbf{U}^{(2)}$$

2nd-order frequency shift:

$$\lambda^{(2)} = \frac{1}{< \mathbf{B} \mathbf{U}^{(0)}; \mathbf{U}^{(0)} >} \left[< \bar{\mathbf{A}} \mathbf{U}^{(1)} + \tilde{\mathbf{A}} \mathbf{U}^{(0)}; \mathbf{U}^{(0)} > - \lambda^{(1)} < \mathbf{B} \mathbf{U}^{(1)}; \mathbf{U}^{(0)} > \right. \\ \left. + \int_{S_T} (\bar{\mathbf{K}} \mathbf{U}^{(1)} + \tilde{\mathbf{K}} \mathbf{U}^{(0)})_{M\alpha} N_M u_\alpha^{(0)} dS + \int_{S_D} (\bar{\mathbf{D}} \mathbf{U}^{(1)} + \tilde{\mathbf{D}} \mathbf{U}^{(0)})_L N_L \phi^{(0)} dS \right]$$

(J. S. Yang and J. A. Kosinski, IEEE TUFFC, 53, 2442-2449, 2006)

有简并时的摄动

Eigenvalue problem: $(\mathbf{A} + \varepsilon \bar{\mathbf{A}}) \mathbf{U} = \lambda \mathbf{B} \mathbf{U}$ Perturbation expansions: $\lambda \equiv \lambda^{(0)} + \varepsilon \lambda^{(1)}$

$$\mathbf{U} \equiv \sum_{n=1}^N \gamma_{(n)} \mathbf{U}_{(n)}^{(0)} + \varepsilon \mathbf{U}^{(1)}$$

Frequency shifts: $\sum_{n=1}^N (A_{(mn)} - \lambda^{(1)} B_{(mn)}) \gamma_{(n)} = 0$

where $A_{(mn)} = < \bar{\mathbf{A}} \mathbf{U}_{(n)}^{(0)}; \mathbf{U}_{(m)}^{(0)} > + \int_{S_T} (\bar{\mathbf{K}} \mathbf{U}_{(n)}^{(0)})_{M\alpha} N_M u_{\alpha(M)}^{(0)} dS + \int_{S_D} (\bar{\mathbf{D}} \mathbf{U}_{(n)}^{(0)})_L N_L \phi_{(m)}^{(0)} dS$

 $= \int_V (\bar{c}_{L,M\alpha} u_{\alpha(M),L}^{(0)} u_{\gamma(m),L}^{(0)} + \bar{e}_{ML\gamma} u_{\gamma(m),L}^{(0)} \phi_{(m),M}^{(0)} - \bar{e}_{KL} \phi_{(m),L}^{(0)} \phi_{(n),K}^{(0)}) dV = A_{(nm)}$
 $B_{(mn)} = < \mathbf{B} \mathbf{U}_{(n)}^{(0)}; \mathbf{U}_{(m)}^{(0)} > + \int_V \rho_0 u_{\alpha(m)}^{(0)} u_{\alpha(n)}^{(0)} dV = B_{(nm)}$

问题: 特征值密集分布时的摄动方法?

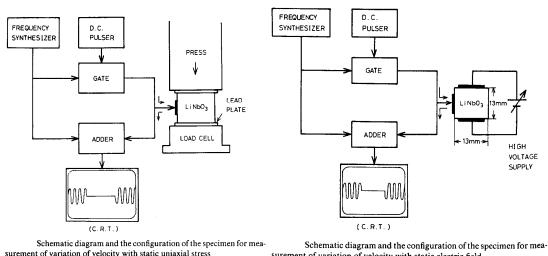
(J. S. Yang and J. A. Kosinski, IEEE TUFFC, 53, 2442-2449, 2006)

梁、板、壳理论

- Y. T. Hu, J. S. Yang and Q. Jiang, Characterization of electroelastic beams under biasing fields with applications in buckling analysis, Archive of Applied Mechanics, 72, 439-450, 2002.
- Y. T. Hu, J. S. Yang and Q. Jiang, A model of electroelastic plates under biasing fields with applications in buckling analysis, International Journal of Solids and Structures, 39, 2629-2642, 2002.
- Y. T. Hu, J. S. Yang and Q. Jiang, On modeling of extension and flexure response of electroelastic shells under biasing fields, Acta Mechanica, 156, 163-178, 2002.

非线性材料常数的测定

Pulse-echo: under a mechanical or electric bias



Thank You