国家基金委压电声波器件分析理论与方法暑期讲习班
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Linear and Nonlinear Analyses of Piezoelectric Plates

压电板的线性和非线性分析

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Outline

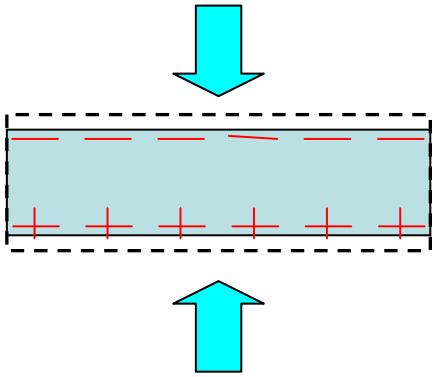
- Theory of piezoelectricity: A brief review
- Mindlin's theory for piezoelectric plates
- Different treatments for linear plate analysis
- Nonlinear plate analysis
- Concluding remarks

Outline

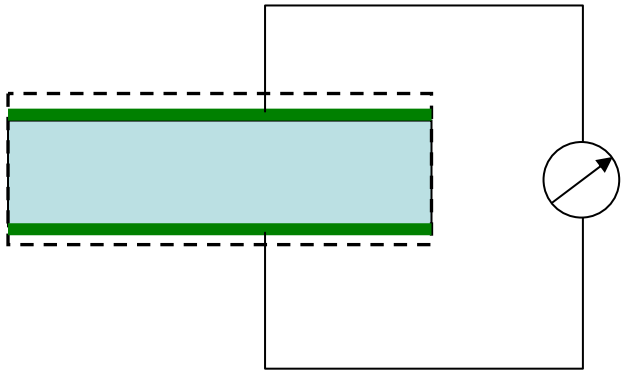
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Piezoelectricity

- **Direct and converse piezoelectric effects**



Direct piezoelectric effect



Converse piezoelectric effect

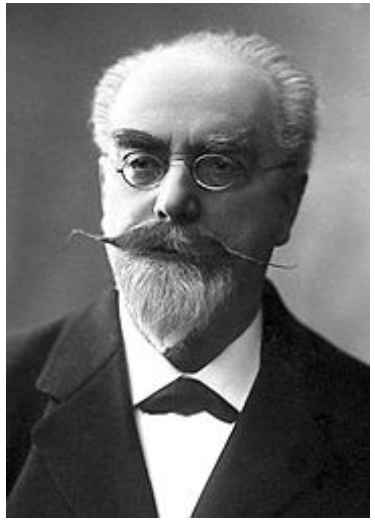
- **Applications: wristwatch, clock, cell phone, sensor, actuator, MEMS, ….**

History – Piezoelectricity

- **Curie (Jacques and Pierre) brothers:**
  Experimental discovery of direct piezoelectric effect (1880) and confirmation of converse piezoelectric effect (1882)



Curie brothers and their parents



G. Lippmann (1845-1921)



W. Voigt (1850-1919)

- **Gabriel Lippmann:** **Theoretical prediction of converse piezoelectric effect (1881)**
- **W. Voigt:** **General thermodynamic theory (1890-1894)**

References

- Cady WG. (1964) Piezoelectricity: An Introduction to the Theory and Applications of Electromechanical Phenomena. New York: Dover.
- Tiersten HF (1969) Linear Piezoelectric Plate Vibrations. New York: Plenum.
- Nelson DF (1979) Electric, Optic and Acoustic Interactions in Crystals. New York: John Wiley & Sons.
- Yang JS (2005) An Introduction to the Theory of Piezoelectricity. New York: Springer.
- Yang JS (2006) Analysis of Piezoelectric Devices. Singapore: World Scientific.

Linear Theory – 1

Balance equations

The equations of motion (Cauchy's first and second laws):

$$T_{ji,j} + f_i = \rho \ddot{u}_i, \qquad T_{ij} = T_{ji}$$

The equilibrium of electrostatics (Gaussian equation)

$$D_{i,i} = \rho_e$$

Ding HJ, Chen WQ (2001) Three Dimensional Problems of Piezoelasticity. New York: Nova Science Publishers.

Linear Theory – 2

■ Constitutive equations

(I) $S_{ij} = s^{E}_{ijkl} T_{kl} + d_{kij} E_k \qquad D_i = d_{ikl} T_{kl} + \varepsilon^{T}_{ik} E_k$

(II) $T_{ij} = c^{E}_{ijkl} S_{kl} - e_{kij} E_k \qquad D_i = e_{ikl} S_{kl} + \varepsilon^{S}_{ik} E_k$

(III) $S_{ij} = s^{D}_{ijkl} T_{kl} + g_{kij} D_k \qquad E_i = -g_{ikl} T_{kl} + \beta^{T}_{ik} D_k$

(IV) $T_{ij} = c^{D}_{ijkl} S_{kl} - h_{kij} D_k \qquad E_i = -h_{ikl} S_{kl} + \beta^{S}_{ik} D_k$

Linear Theory – 3

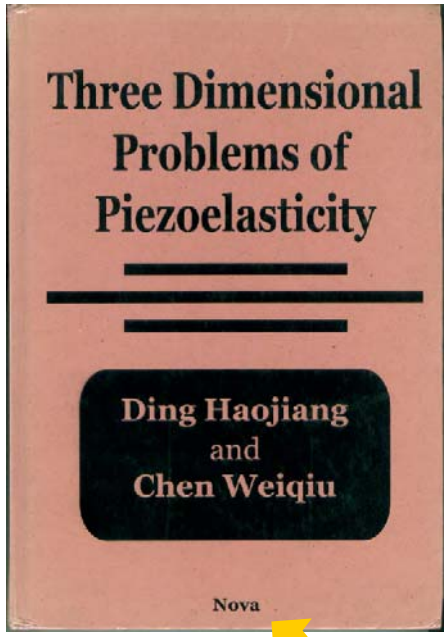
■ Gradient relations

Strain-displacement relations

$$S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Electric field intensity-electric potential relation

$$E_i = \phi_{,i}$$



The three sets of equations should be supplemented with appropriate boundary and initial conditions to form a well-defined boundary value problem in mathematics.

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- Theory of piezoelectricity: A brief review
- **Mindlin's theory for piezoelectric plates**
- Different treatments for linear plate analysis
- Nonlinear plate analysis
- Concluding remarks

Development of Plate Theory

- S. Germain, S. D. Poisson, J. L. Lagrange, G. Kirchhoff, E. Reissner
- R. D. Mindlin
- H. F. Tiersten
- P. C. Y. Lee
- Y. K. Yong, J. S. Yang, J. Wang



R.D. Mindlin (1906-1987)

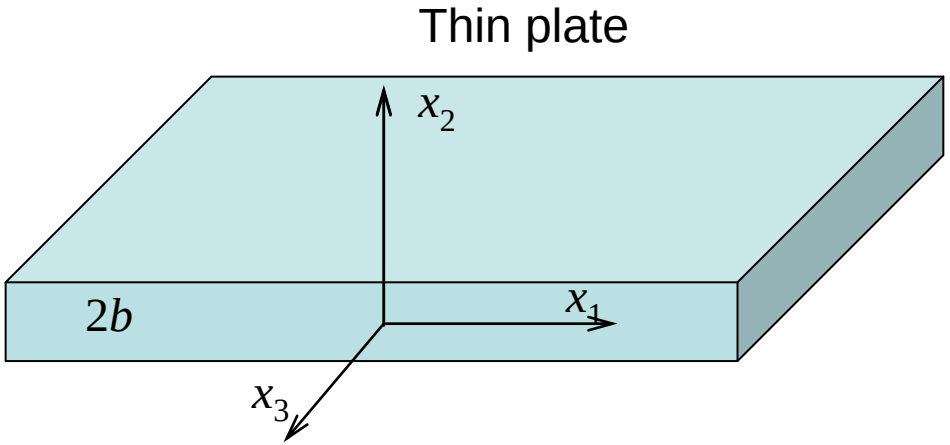


Mindlin's Theory – 1

■ Power series expansion

$$u_i(x_1, x_2, x_3, t) = \sum_{n=0}^{\infty} x_2^n u_i^{(n)}(x_1, x_3, t)$$

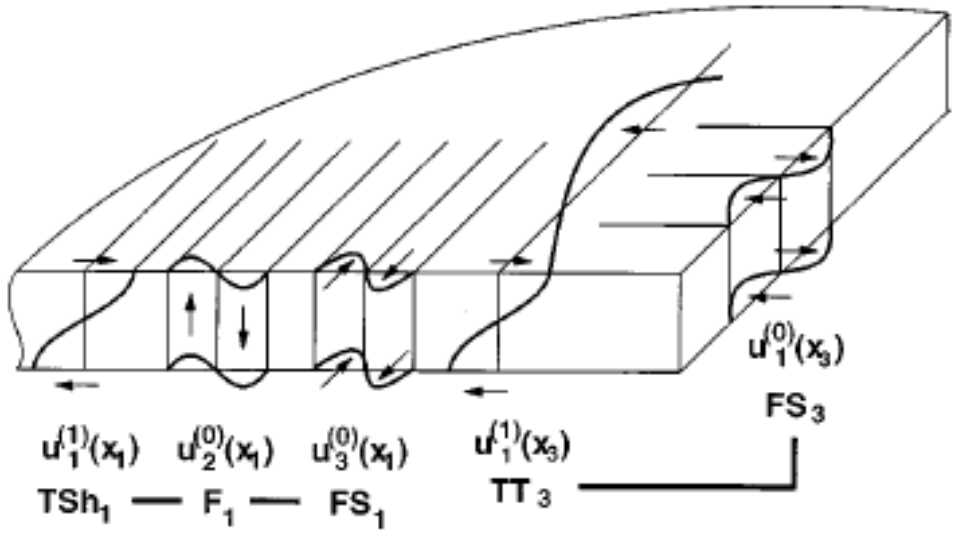
$$\phi(x_1, x_2, x_3, t) = \sum_{n=0}^{\infty} x_2^n \phi^{(n)}(x_1, x_3, t)$$



$u_1^{(0)},\ u_3^{(0)}$	⟹	In-plane stretch and shear (Face shear)
$u_2^{(0)}$	⟹	Flexure
$u_1^{(1)},\ u_3^{(1)}$	⟹	Fundamental thickness-shear/twist
$u_2^{(1)}$	⟹	Fundamental thickness-stretch
$u_1^{(2)}, u_3^{(2)}$	⟹	Symmetric thickness-shear/twist

Mindlin's Theory – 2

- **Fundamental modes**



Mode shapes of thickness shear, flexural and face shear varying in x_1 direction and thickness twist and face shear varying in x_3 direction

Mindlin's Theory – 3

Basic formulations

Strains and electric fields

$$S_{ij} = \sum_n x_2^n S_{ij}^{(n)}$$

$$E_i = \sum_n x_2^n E_i^{(n)}$$

$$S_{ij}^{(n)} = \frac{1}{2}[u_{j,i}^{(n)} + u_{i,j}^{(n)} + (n+1)(\delta_{i2}u_j^{(n+1)} + \delta_{2j}u_i^{(n+1)}]$$

$$E_i^{(n)} = -\phi_{,i}^{(n)} - (n+1)\delta_{2i}\phi^{(n+1)}$$

Constitutive relations

$$T_{ij}^{(n)} = \sum_m B_{mn}(c_{ijkl}S_{kl}^{(m)} - e_{kij}E_k^{(m)}),$$

$$D_i^{(n)} = \sum_m B_{mn}(e_{ijk}S_{jk}^{(m)} - \varepsilon_{ij}E_j^{(m)}),$$

$$T_{ij}^{(n)} = \int_{-b}^{b} T_{ij}x_2^n \,\mathrm{d}\,x_2$$

$$D_i^{(n)} = \int_{-b}^{b} D_i x_2^n \,\mathrm{d}\,x_2$$

$$B_{mn} = \int_{-h}^{h} x_2^m x_2^n \,\mathrm{d}\,x_2 = \begin{cases} 2b^{m+n+1}/(m+n+1), & m+n \quad \text{even} \\ 0, & m+n \quad \text{odd} \end{cases}$$

Mindlin's Theory – 3

Basic formulations (cont.)

Equations of motion and charge

$$T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + F_j^{(n)} = \rho \sum_m B_{mn} \ddot{u}_j^{(m)}$$

$$D_{i,i}^{(n)} - nD_2^{(n-1)} + D^{(n)} = 0$$

$$F_j^{(n)} = [x_2^n T_{2j}]_{-b}^{b} + \int_{-b}^{b} \rho f_j x_2^n \, \mathrm{d}x_2$$

$$D^{(n)} = [x_2^n D_2]_{-b}^{b} - \int_{-b}^{b} \rho_e x_2^n \, \mathrm{d}x_b$$

Boundary conditions at the cylindrical boundary

$$T_{aj}^{(n)} n_a = \bar{t}_j^{(n)}$$

$$D_a^{(n)} n_a = -\bar{\sigma}_e^{(n)}$$

$$\bar{t}_j^{(n)} = \int_{-b}^{b} \bar{t}_j x_2^n \, \mathrm{d}x_2 \qquad \bar{\sigma}_e^{(n)} = \int_{-b}^{b} \bar{\sigma}_e x_2^n \, \mathrm{d}x_2$$

or

$$u_j^{(n)} = \bar{u}_j^{(n)}$$

$$\phi^{(n)} = \bar{\phi}^{(n)}$$

Mindlin's Theory – 4

■ Truncation and adjustment

The process of truncation of the series and adjustment of the remaining terms begins with the discard of the strains and electric fields of order higher than the first, leaving $S_{ij}^{(0)}$, $S_{ij}^{(1)}$, $E_i^{(0)}$, $E_i^{(1)}$ which contain the zero, first and second order mechanical displacements, $u_i^{(0)}$, $u_i^{(1)}$, $u_j^{(2)}$ and electric potentials, $\varphi^{(0)}$, $\varphi^{(1)}$, $\varphi^{(2)}$, some of which will be eliminated subsequently.

(1) Neglect inertial terms associated with $u_2^{(1)}$ and $u_j^{(2)}$;

(2) Set $T_{22}^{(0)} = 0$ to eliminate $u_2^{(1)}$;

(3) Assume $\phi^{(2)} = 0$;

(4) Set $T_{2j}^{(1)} = 0$ to eliminate $u_j^{(2)}$;

(5) Adjust by introducing shear correction factors $S_{21}^{(0)} \to \kappa_1 S_{21}^{(0)}$, $S_{23}^{(0)} \to \kappa_3 S_{23}^{(0)}$.

Mindlin RD (1972) High frequency vibrations of piezoelectric crystal plates. International Journal of Solids and Structures, **8**: 895-906.

Mindlin's Theory – 5

First-order plate theory

Coupled extensional, flexural and thickness-shear motions

$$u_i \cong u_i^{(0)} + x_2 u_i^{(1)} + x_2^2 u_i^{(2)}$$
$$\phi \cong \phi^{(0)} + x_2 \phi^{(1)}$$

$$T_{ab,a}^{(0)} + F_b^{(0)} = 2b\rho \ddot{u}_b^{(0)}, \qquad a,b = 1,3$$
$$T_{a2,a}^{(0)} + F_2^{(0)} = 2b\rho \ddot{u}_2^{(0)},$$
$$T_{ab,a}^{(1)} - T_{2b}^{(0)} + F_b^{(1)} = \frac{2b^3}{3} \rho \ddot{u}_b^{(1)},$$
$$D_{a,a}^{(0)} + D^{(0)} = 0,$$
$$D_{a,a}^{(1)} - D_2^{(0)} + D^{(1)} = 0,$$

Wang J, Yang JS (2000) Higher-order theories of piezoelectric plates and applications. Applied Mechanics Reviews, **53**: 87-99.

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“Displacement” Equation

- **Balance equations in terms of displacements and electric potential**

$$c_{ijkl}u_{k,li} + e_{kij}\phi_{,ki} = \rho\ddot{u}_j$$

$$e_{ikl}u_{k,li} - \varepsilon_{ik}\phi_{,ki} = 0$$

$$u_i(x_1, x_2, x_3, t) = \sum_{n=0}^{\infty} x_3^n u_i^{(n)}(x_1, x_2, t)$$

$$\phi(x_1, x_2, x_3, t) = \sum_{n=0}^{\infty} x_3^n \phi^{(n)}(x_1, x_2, t)$$

Observation: If the displacements and electric potential are expanded in terms of power series of the thickness coordinate, then these equations rightly relate the coefficients of high-order to those of low-order!

Boström A, Johansson G, Olsson P (2001) On the rational derivation of a hierarchy of dynamic equations for a homogeneous, isotropic, elastic plate. International journal of Solids and Structures, **38**: 2487-2501.

Mauritsson K, Boström A, Folkow PD (2008) Modelling of thin piezoelectric layers on plates. Wave Motion, 45: 616-628.

Mauritsson K, Folkow PD, Boström A (2011) Dynamic equations for a fully anisotropic elastic plate. Journal of Sound and Vibration, **330**: 2640-2654.

BJO Treatment – 1

- **Recursive relations (for class 2mm)**

$$u_1^{(n+2)} = \frac{1}{(n+1)(n+2)c_{55}}\left\{\rho\ddot{u}_1^{(n)} - c_{11}u_{1,11}^{(n)} - c_{66}u_{1,22}^{(n)} - c_{126}u_{2,12}^{(n)} - (n+1)\left[c_{135}u_{3,1}^{(n+1)} + e_{51}\phi_{,1}^{(n+1)}\right]\right\}$$

$$u_2^{(n+2)} = \frac{1}{(n+1)(n+2)c_{44}}\left\{\rho\ddot{u}_2^{(n)} - c_{22}u_{2,22}^{(n)} - c_{66}u_{2,11}^{(n)} - c_{126}u_{1,12}^{(n)} - (n+1)\left[c_{234}u_{3,2}^{(n+1)} - e_{42}\phi_{,2}^{(n+1)}\right]\right\}$$

$$u_3^{(n+2)} = \frac{1}{(n+1)(n+2)h_{33}}\left\{\varepsilon_{33}\rho\ddot{u}_3^{(n)} - h_{55}u_{3,11}^{(n)} - h_{44}u_{3,22}^{(n)} - (n+1)\left[h_{135}u_{1,1}^{(n+1)} + h_{234}u_{2,2}^{(n+1)}\right] - f_{53}\phi_{,11}^{(n)} - f_{43}\phi_{,22}^{(n)}\right\}$$

$$\phi^{(n+2)} = \frac{-1}{(n+1)(n+2)h_{33}}\left\{e_{33}\rho\ddot{u}_3^{(n)} - g_{53}u_{3,11}^{(n)} - g_{43}u_{3,22}^{(n)} - (n+1)\left[g_{135}u_{1,1}^{(n+1)} + g_{234}u_{2,2}^{(n+1)}\right] - h_{35}\phi_{,11}^{(n)} - h_{34}\phi_{,22}^{(n)}\right\}$$

- **Substitution into boundary conditions at the upper/lower surfaces gives rise to eight equations about eight functions** $u_i^{(0)}, u_i^{(1)}, \phi^{(0)}, \phi^{(1)}$

BJO Treatment – 2

- **Truncation:** **The yielded eight plate equations are expressed as an infinite power series of plate thickness, and should be truncated for practical problem solving.**

  (1) Transform the equations into a single equation, and then discard all terms higher than a certain order. (BJO, 2001)

  (2) Discard in each equation all terms higher than a certain order; (MFB, 2011)

$$\Delta_{11}^{(0)}u+\Delta_{12}^{(0)}w+(\Delta_{11}^{(2)}u+\Delta_{12}^{(2)}w)h^2+(\Delta_{11}^{(4)}u+\Delta_{12}^{(4)}w)h^4+\cdots=0$$
$$(\Delta_{21}^{(1)}u+\Delta_{22}^{(1)}w)h+(\Delta_{21}^{(3)}u+\Delta_{22}^{(3)}w)h^3+(\Delta_{21}^{(5)}u+\Delta_{22}^{(5)}w)h^5+\cdots=0$$

$$\Delta_{11}^{(0)}u+\Delta_{12}^{(0)}w+(\Delta_{11}^{(2)}u+\Delta_{12}^{(2)}w)h^2=0$$
$$(\Delta_{21}^{(1)}u+\Delta_{22}^{(1)}w)h=0$$

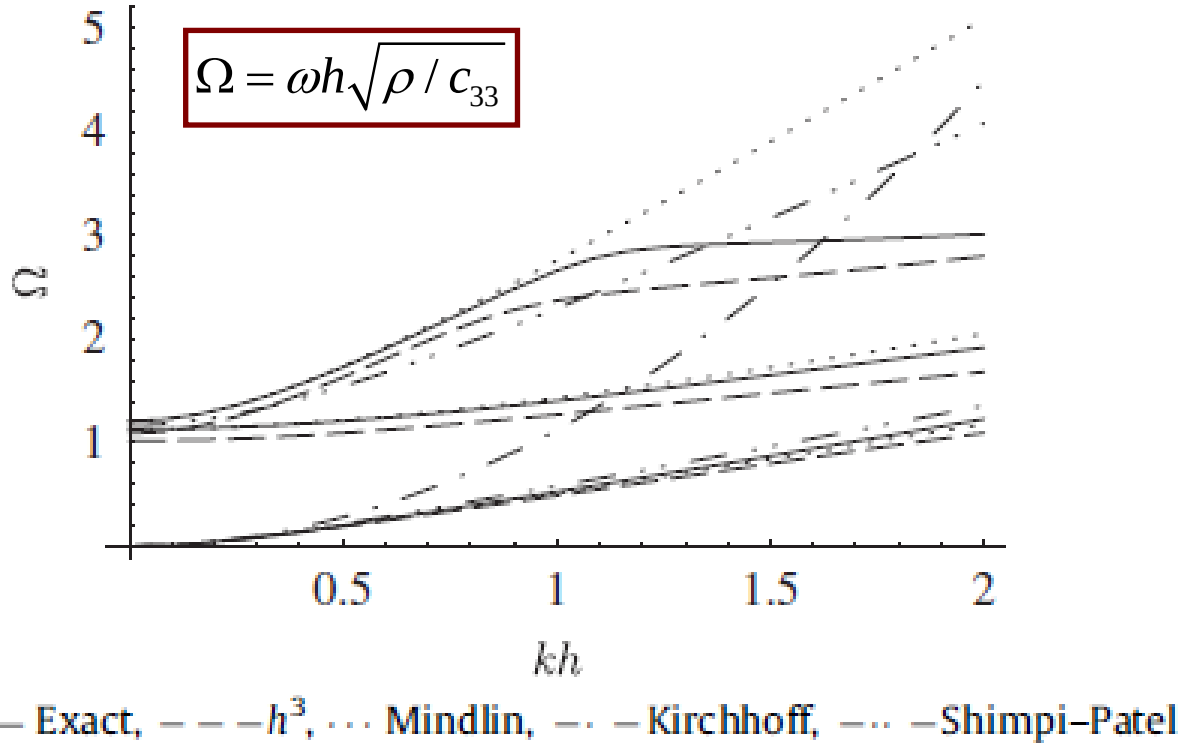
$$(\Delta_{22}^{(1)}\Delta_{11}^{(0)}-\Delta_{12}^{(0)}\Delta_{21}^{(1)})hu=0$$

Numerical results – 1

■ Elastic plate of thickness 2h

Graphite-epoxy (AS/3501)

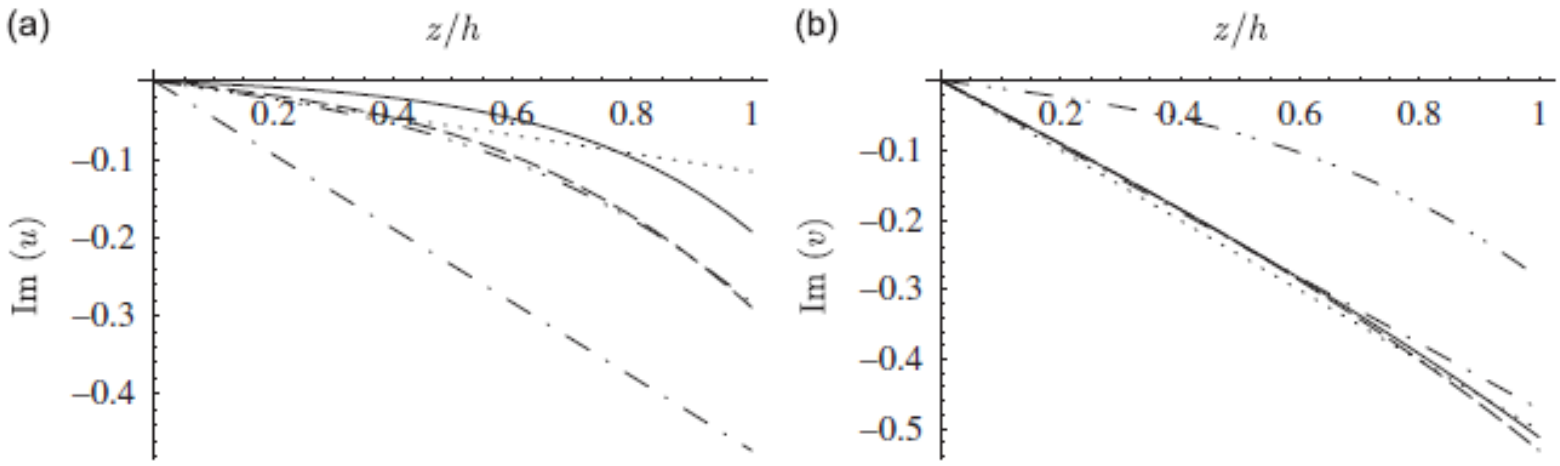
$c_{11} = 141.1,\ c_{12} = c_{13} = 5.4,$
$c_{22} = c_{33} = 12,\ c_{23} = 6.0,$
$c_{44} = 6.2,\ c_{55} = c_{66} = 7.1\ \text{GPa}$



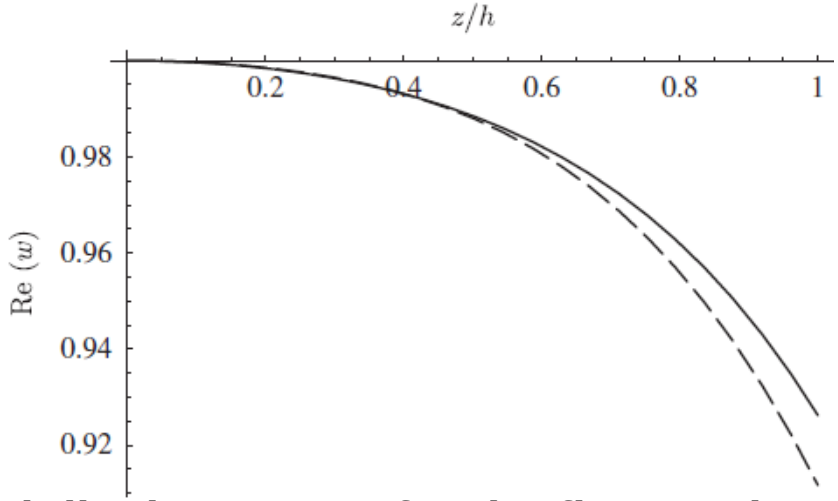
Dispersion curves for waves propagating in a direction 45°

Mauritsson K, Folkow PD, Boström A (2011) Dynamic equations for a fully anisotropic elastic plate. Journal of Sound and Vibration, **330**: 2640-2654.

Numerical results – 2

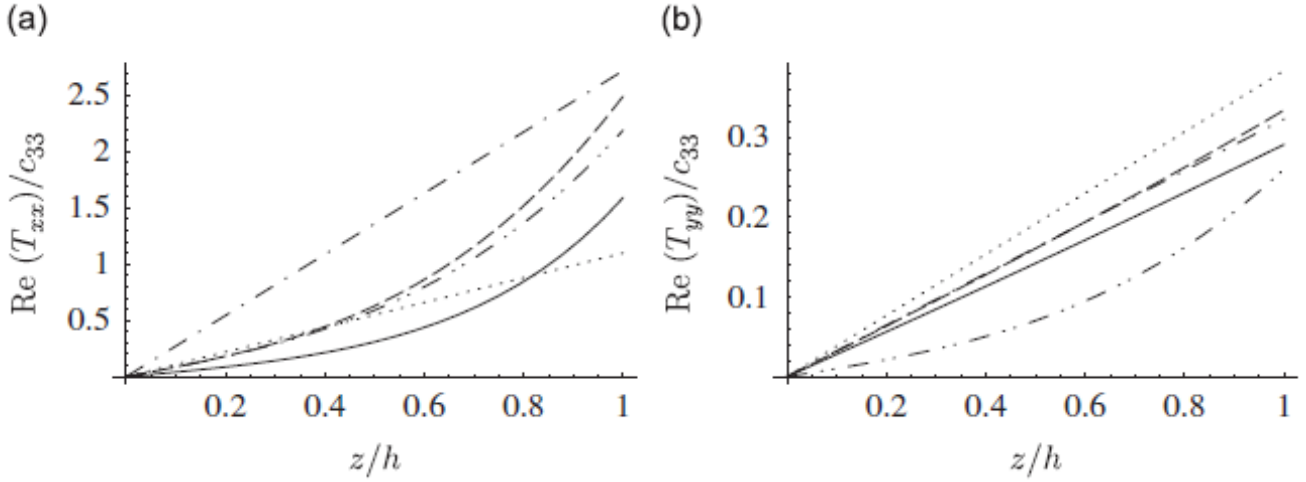


Horizontal displacements in the (a) x direction and (b) y direction for the first mode when Ω=0.5.

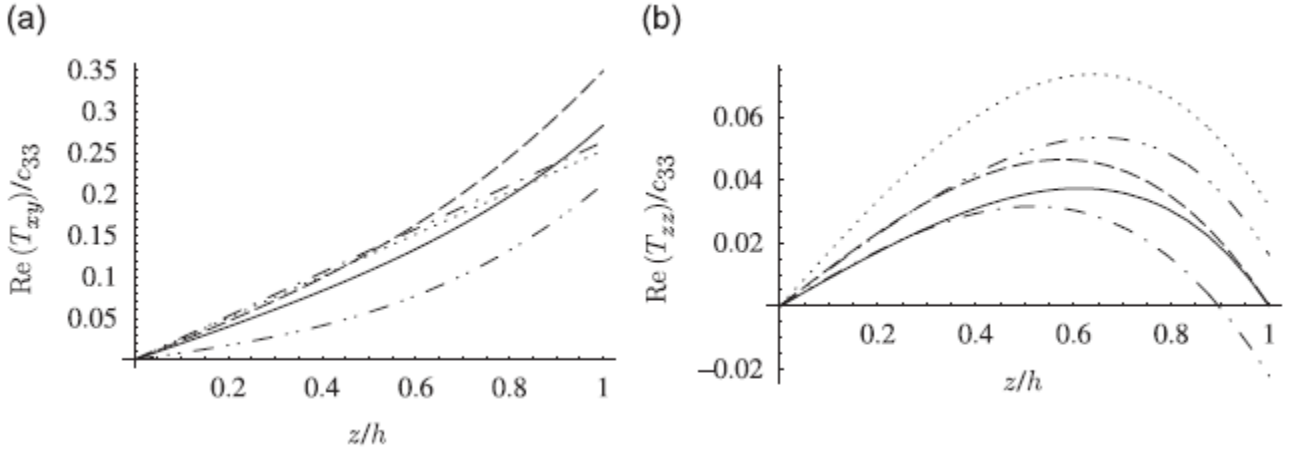


Vertical displacements for the first mode when Ω=0.5.

Numerical results – 3

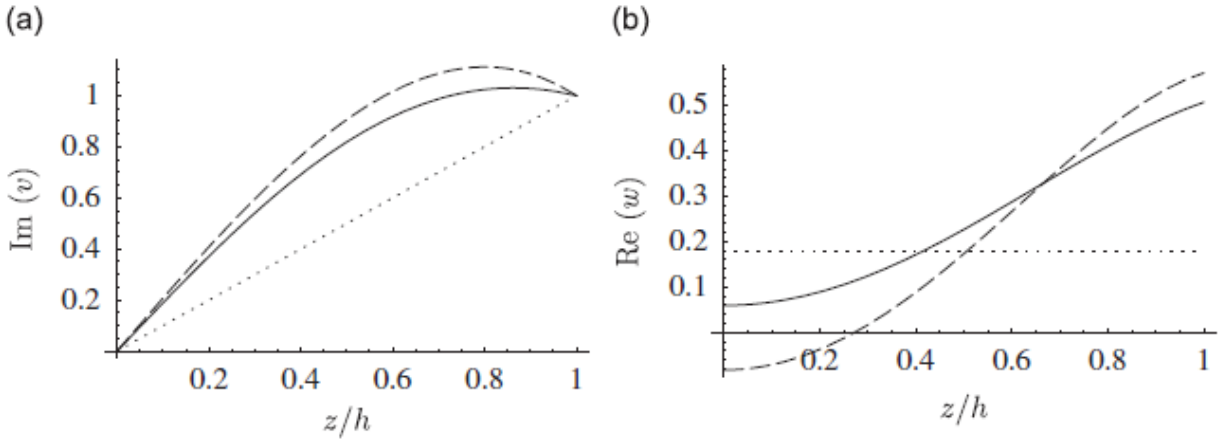


Normal stresses in the (a) x direction and (b) y direction for the first mode when Ω=0.5.

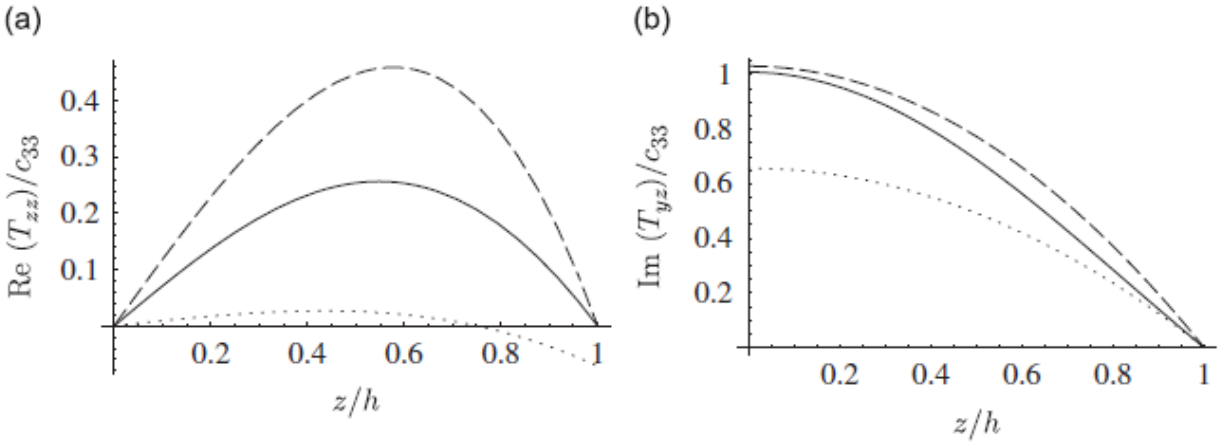


Stresses (a) T_{xy} and (b) T_{zz} for the first mode when Ω=0.5.

Numerical results – 4



Displacements in the (a) x direction and (b) z direction for the second mode when Ω=1.5.

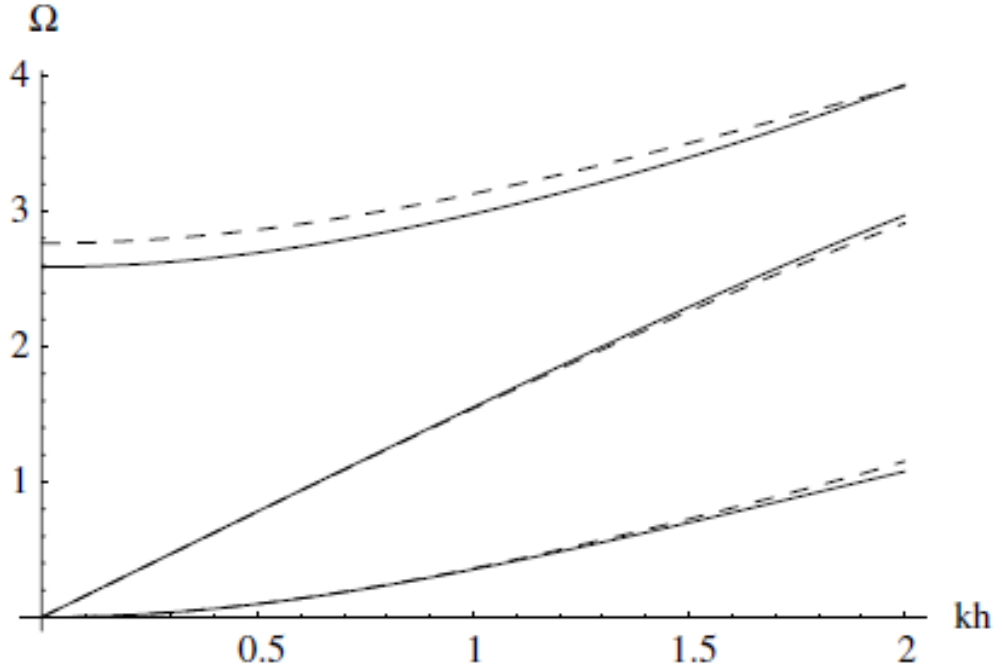


Stresses (a) T_{zz} and (b) T_{yz} for the second mode when Ω=1.5.

Numerical results – 5

- **A PZT-2 (of 6mm) layer on a steel plate**

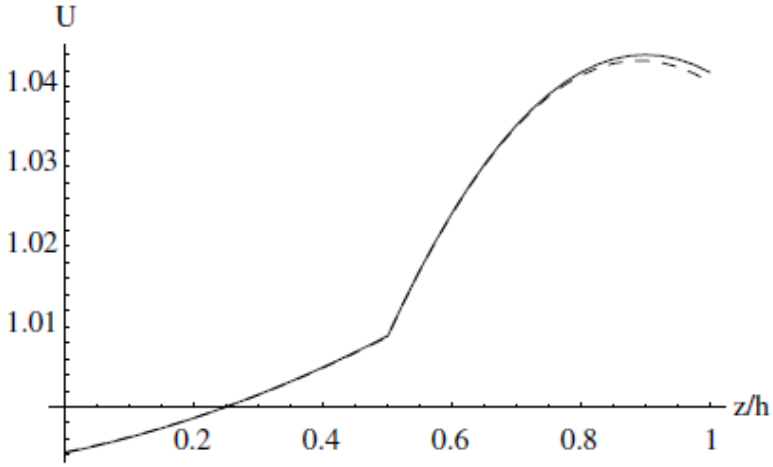
$$\Omega = \omega h \sqrt{\rho_e / c_{55}}$$
$$(h = h_e + h_p)$$
$$\alpha = h_p / h_e$$



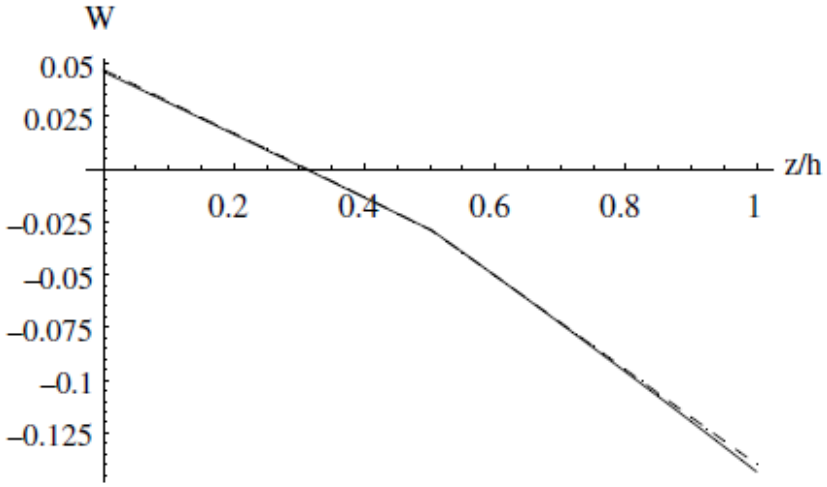
Dispersion curves (α = 1/3): solid – approximate, dashed – exact.

Mauritsson K, Boström A, Folkow PD (2008) Modelling of thin piezoelectric layers on plates. Wave Motion, 45: 616-628.

Numerical results – 6

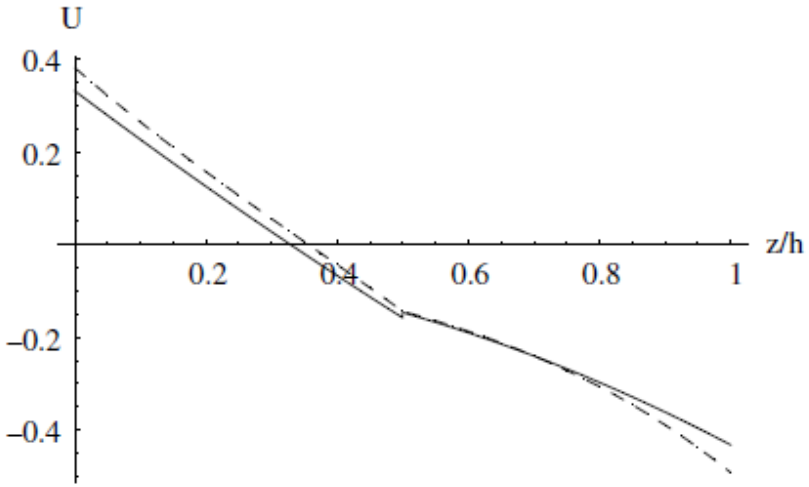


Horizontal displacement distribution for the first in-plane mode (α = 1, Ω=0.5)

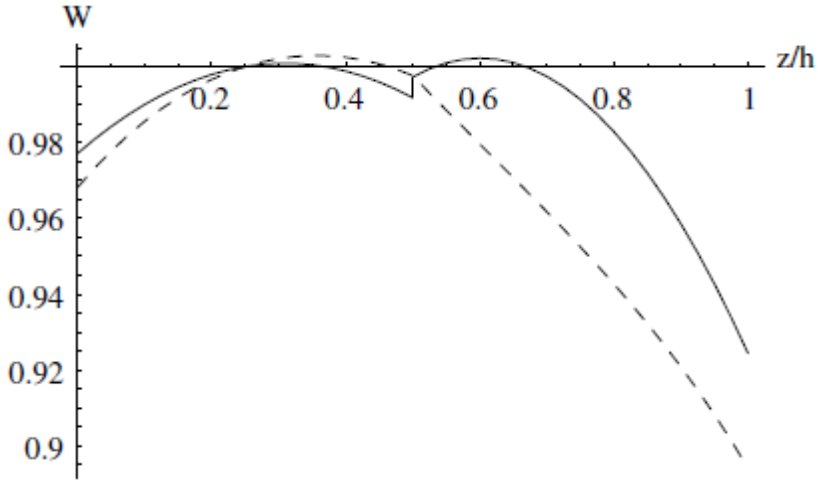


Vertical displacement distribution for the first in-plane mode (α = 1, Ω=0.5)

Numerical results – 7



Horizontal displacement distribution for the first bending mode (α = 1, Ω=0.5)



Vertical displacement distribution for the first bending mode (α = 1, Ω=0.5)

Our new treatment A – 1

■ Piezoelectric plate (of 6mm)

"Displacement" governing equations

$$\left(c_{11}\frac{\partial^2}{\partial x^2}+c_{66}\frac{\partial^2}{\partial y^2}+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)u_x+(c_{12}+c_{66})\frac{\partial^2 u_y}{\partial x\partial y}+(c_{13}+c_{44})\frac{\partial^2 u_z}{\partial x\partial z}+(e_{15}+e_{31})\frac{\partial^2\phi}{\partial x\partial z}=0$$

$$(c_{12}+c_{66})\frac{\partial^2 u_x}{\partial x\partial y}+\left(c_{66}\frac{\partial^2}{\partial x^2}+c_{11}\frac{\partial^2}{\partial y^2}+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)u_y+(c_{13}+c_{44})\frac{\partial^2 u_z}{\partial y\partial z}+(e_{15}+e_{31})\frac{\partial^2\phi}{\partial y\partial z}=0$$

$$(c_{13}+c_{44})\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)+\left(c_{44}\Lambda+c_{33}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)u_z+\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right)\phi=0$$

$$(e_{15}+e_{31})\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right)+\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right)u_z-\left(\varepsilon_{11}\Lambda+\varepsilon_{33}\frac{\partial^2}{\partial z^2}\right)\phi=0$$

Our new treatment A – 2

- **Piezoelectric plate (of 6mm) (cont.)**

  Boundary conditions

$$T_{xz} = c_{44}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + e_{15}\frac{\partial \phi}{\partial x} = 0$$

$$T_{yz} = c_{44}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) + e_{15}\frac{\partial \phi}{\partial y} = 0$$

$$T_{zz} = c_{13}\frac{\partial u_x}{\partial x} + c_{13}\frac{\partial u_y}{\partial y} + c_{33}\frac{\partial u_z}{\partial z} + e_{33}\frac{\partial \phi}{\partial z} = 0 \qquad (\text{at } z = \pm h)$$

$$D_z = e_{31}\frac{\partial u_x}{\partial x} + e_{31}\frac{\partial u_y}{\partial y} + e_{33}\frac{\partial u_z}{\partial z} - \varepsilon_{33}\frac{\partial \Phi}{\partial z} = 0$$

Our new treatment A – 3

■ Nondimensionalization

"Displacement" governing equations

$$\left(c_{11}^0\frac{\partial^2}{\partial\xi^2}+c_{66}^0\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\zeta^2}-\frac{\partial^2}{\partial\tau^2}\right)U+(c_{12}^0+c_{66}^0)\frac{\partial^2V}{\partial\xi\partial\eta}+(c_{13}^0+c_{44}^0)\frac{\partial^2W}{\partial\xi\partial\zeta}+(e_{15}^0+e_{31}^0)f^0\frac{\partial^2\Phi}{\partial\xi\partial\zeta}=0$$

$$(c_{12}^0+c_{66}^0)\frac{\partial^2U}{\partial\xi\partial\eta}+\left(c_{66}^0\frac{\partial^2}{\partial\xi^2}+c_{11}^0\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\zeta^2}-\frac{\partial^2}{\partial\tau^2}\right)V+(c_{13}^0+c_{44}^0)\frac{\partial^2W}{\partial\eta\partial\zeta}+(e_{15}^0+e_{31}^0)f^0\frac{\partial^2\Phi}{\partial\eta\partial\zeta}=0$$

$$(c_{13}^0+c_{44}^0)\frac{\partial}{\partial\zeta}\left(\frac{\partial U}{\partial\xi}+\frac{\partial V}{\partial\eta}\right)+\left(\Lambda^0+c_{33}^0\frac{\partial^2}{\partial\zeta^2}-\frac{\partial^2}{\partial\tau^2}\right)W+\left(e_{15}^0f^0\Lambda^0+f^0\frac{\partial^2}{\partial\zeta^2}\right)\Phi=0$$

$$(e_{15}^0+e_{31}^0)\frac{\partial}{\partial\zeta}\left(\frac{\partial U}{\partial\xi}+\frac{\partial V}{\partial\eta}\right)+\left(e_{15}^0\Lambda^0+\frac{\partial^2}{\partial\zeta^2}\right)W-\left(\varepsilon_{11}^0\varepsilon_c f^0\Lambda^0+\varepsilon_c f^0\frac{\partial^2}{\partial\zeta^2}\right)\Phi=0$$

$$u_x=dU,\ u_y=dV,\ u_z=dW,\ \phi=f\Phi,\ x=l\xi,\ y=l\eta,\ z=l\zeta,\ t=\tau l/c$$

$$c_{ij}^0=c_{ij}/c_{44},\ e_{ij}^0=e_{ij}/e_{33},\ \varepsilon_{ij}^0=\varepsilon_{ij}/\varepsilon_{33},\ f^0=e_{33}f/(c_{44}d),\ \varepsilon_c=\varepsilon_{33}c_{44}/e_{33}^2$$

Our new treatment A – 4

■ Nondimensionalization (cont.)

Boundary conditions

$$\frac{\partial U}{\partial \zeta}+\frac{\partial W}{\partial \xi}+e_{15}^{0} f^{0} \frac{\partial \Phi}{\partial \xi}=0$$

$$\frac{\partial V}{\partial \zeta}+\frac{\partial W}{\partial \eta}+e_{15}^{0} f^{0} \frac{\partial \Phi}{\partial \eta}=0$$

$$c_{13}^{0} \frac{\partial U}{\partial \xi}+c_{13}^{0} \frac{\partial V}{\partial \eta}+c_{33}^{0} \frac{\partial W}{\partial \zeta}+ f^{0} \frac{\partial \Phi}{\partial \zeta}=0$$

$$e_{31}^{0} \frac{\partial U}{\partial \xi}+e_{31}^{0} \frac{\partial V}{\partial \eta}+\frac{\partial W}{\partial \zeta}- g^{0} \frac{\partial \Phi}{\partial \zeta}=0$$

$$g^{0}=\varepsilon_{c} f^{0}$$

$(\text{at } \zeta=\pm\kappa)$

$\kappa = h/l$ is a small parameter!

Our new treatment A – 5

- **Series expansions**

$$U = \sum_{n=0}^{\infty} \zeta^n U_n(\xi, \eta, \tau; \delta, \kappa),\ V = \sum_{n=0}^{\infty} \zeta^n V_n(\xi, \eta, \tau; \delta, \kappa)$$

$$W = \sum_{n=0}^{\infty} \zeta^n U_n(\xi, \eta, \tau; \delta, \kappa), \Phi = \sum_{n=0}^{\infty} \zeta^n \Phi_n(\xi, \eta, \tau; \delta, \kappa)$$

$\delta = d/l$ is another small parameter!

Dai HH, Huo Y (2002) Asymptotically approximate model equations for nonlinear dispersive waves in incompressible elastic rods, Acta Mechanica, **157**: 97-112.

Our new treatment A – 6

■ Series expansions (cont.)

Governing equations

$$c_{134}^0 = c_{13}^0 + c_{44}^0$$
$$c_{126}^0 = c_{12}^0 + c_{66}^0$$
$$e_{151}^0 = e_{15}^0 + e_{31}^0$$

$$c_{11}^0 \frac{\partial^2 U_n}{\partial \xi^2} + c_{66}^0 \frac{\partial^2 U_n}{\partial \eta^2} + (n+2)(n+1)U_{n+2} - \frac{\partial^2 U_n}{\partial \tau^2} + c_{126}^0 \frac{\partial^2 V_n}{\partial \xi \partial \eta}$$
$$+ c_{134}^0 (n+1)\frac{\partial W_{n+1}}{\partial \xi} + e_{151}^0 f^0 (n+1)\frac{\partial \Phi_{n+1}}{\partial \xi} = 0$$

$$c_{126}^0 \frac{\partial^2 U_n}{\partial \xi \partial \eta} + c_{66}^0 \frac{\partial^2 V_n}{\partial \xi^2} + c_{11}^0 \frac{\partial^2 V_n}{\partial \eta^2} + (n+2)(n+1)V_{n+2} - \frac{\partial^2 V_n}{\partial \tau^2}$$
$$+ c_{134}^0 (n+1)\frac{\partial W_{n+1}}{\partial \eta} + e_{151}^0 f^0 (n+1)\frac{\partial \Phi_{n+1}}{\partial \eta} = 0$$

$$c_{134}^0 (n+1)\left(\frac{\partial U_{n+1}}{\partial \xi} + \frac{\partial V_{n+1}}{\partial \eta}\right) + \Lambda^0 W_n + c_{33}^0 (n+2)(n+1)W_{n+2} - \frac{\partial^2 W_n}{\partial \tau^2}$$
$$+ e_{15}^0 f^0 \Lambda^0 \Phi_n + f^0 (n+2)(n+1)\Phi_{n+2} = 0$$

$$e_{151}^0 (n+1)\left(\frac{\partial U_{n+1}}{\partial \xi} + \frac{\partial V_{n+1}}{\partial \eta}\right) + e_{15}^0 \Lambda^0 W_n + (n+2)(n+1)W_{n+2}$$
$$- \varepsilon_{11}^0 g^0 \Lambda^0 \Phi_n + g^0 (n+2)(n+1)\Phi_{n+2} = 0$$

Our new treatment A – 7

■ Series expansions (cont.)

Recursive relations

$$U_{n+2} = \frac{1}{(n+2)(n+1)}\left[\frac{\partial^2 U_n}{\partial\tau^2} - c_{11}^0\frac{\partial^2 U_n}{\partial\xi^2} - c_{66}^0\frac{\partial^2 U_n}{\partial\eta^2} - c_{126}^0\frac{\partial^2 V_n}{\partial\xi\partial\eta} - (n+1)c_{134}^0\frac{\partial W_{n+1}}{\partial\xi} - (n+1)e_{151}^0 f^0\frac{\partial\Phi_{n+1}}{\partial\xi}\right]$$

$$V_{n+2} = \frac{1}{(n+2)(n+1)}\left[\frac{\partial^2 V_n}{\partial\tau^2} - c_{126}^0\frac{\partial^2 U_n}{\partial\xi\partial\eta} - c_{66}^0\frac{\partial^2 V_n}{\partial\xi^2} - c_{11}^0\frac{\partial^2 V_n}{\partial\eta^2} - (n+1)c_{134}^0\frac{\partial W_{n+1}}{\partial\eta} - (n+1)e_{151}^0 f^0\frac{\partial\Phi_{n+1}}{\partial\eta}\right]$$

$$W_{n+2} = \frac{1}{(n+2)(n+1)}\left[\gamma_{11}\frac{\partial^2 W_n}{\partial\tau^2} + (n+1)\gamma_{12}\left(\frac{\partial U_{n+1}}{\partial\xi} + \frac{\partial V_{n+1}}{\partial\eta}\right) + \gamma_{13}\Lambda^0 W_n + \gamma_{14}\Lambda^0\Phi_n\right]$$

$$\Phi_{n+2} = \frac{1}{(n+2)(n+1)}\left[\gamma_{21}\frac{\partial^2 W_n}{\partial\tau^2} + (n+1)\gamma_{22}\left(\frac{\partial U_{n+1}}{\partial\xi} + \frac{\partial V_{n+1}}{\partial\eta}\right) + \gamma_{23}\Lambda^0 W_n + \gamma_{24}\Lambda^0\Phi_n\right]$$

Our new treatment A – 8

Series expansions (cont.)

Boundary conditions

$$\sum_{n=0}^{\infty}(\pm\kappa)^n\left[(n+1)U_{n+1}+\frac{\partial W_n}{\partial\xi}+e_{15}^0 f^0\frac{\partial\Phi_n}{\partial\xi}\right]=0$$

$$\sum_{n=0}^{\infty}(\pm\kappa)^n\left[(n+1)V_{n+1}+\frac{\partial W_n}{\partial\eta}+e_{15}^0 f^0\frac{\partial\Phi_n}{\partial\eta}\right]=0$$

$$\sum_{n=0}^{\infty}(\pm\kappa)^n\left[c_{13}^0\frac{\partial U_n}{\partial\xi}+c_{13}^0\frac{\partial V_n}{\partial\eta}+(n+1)c_{33}^0 W_{n+1}+(n+1)f^0\Phi_{n+1}\right]=0$$

$$\sum_{n=0}^{\infty}(\pm\kappa)^n\left[e_{31}^0\frac{\partial U_n}{\partial\xi}+e_{31}^0\frac{\partial V_n}{\partial\eta}+(n+1)W_{n+1}-(n+1)\varepsilon_c f^0\Phi_{n+1}\right]=0$$

Our new treatment A – 9

- **Two classes of problems**

  Symmetric or generalized plane-stress problem

$$\sum_{n=0}^{\infty} \kappa^{2n+1} \left[(2n+2) U_{2n+2} + \frac{\partial W_{2n+1}}{\partial \xi} + e_{15}^{0} f^{0} \frac{\partial \Phi_{2n+1}}{\partial \xi} \right] = 0$$

$$\sum_{n=0}^{\infty} \kappa^{2n+1} \left[(2n+2) V_{2n+2} + \frac{\partial W_{2n+1}}{\partial \eta} + e_{15}^{0} f^{0} \frac{\partial \Phi_{2n+1}}{\partial \eta} \right] = 0$$

$$\sum_{n=0}^{\infty} \kappa^{2n} \left[c_{13}^{0} \frac{\partial U_{2n}}{\partial \xi} + c_{13}^{0} \frac{\partial V_{2n}}{\partial \eta} + (2n+1) c_{33}^{0} W_{2n+1} + (2n+1) f^{0} \Phi_{2n+1} \right] = 0$$

$$\sum_{n=0}^{\infty} \kappa^{2n} \left[e_{31}^{0} \frac{\partial U_{2n}}{\partial \xi} + e_{31}^{0} \frac{\partial V_{2n}}{\partial \eta} + (2n+1) W_{2n+1} - (2n+1) \varepsilon_{c} f^{0} \Phi_{2n+1} \right] = 0$$

Our new treatment A – 10

- **Two classes of problems (cont.)**

  Antisymmetric or bending problem

$$\sum_{n=0}^{\infty} \kappa^{2n} \left[(2n+1) U_{2n+1} + \frac{\partial W_{2n}}{\partial \xi} + e_{15}^{0} f^{0} \frac{\partial \Phi_{2n}}{\partial \xi} \right] = 0$$

$$\sum_{n=0}^{\infty} \kappa^{2n} \left[(2n+1) V_{2n+1} + \frac{\partial W_{2n}}{\partial \eta} + e_{15}^{0} f^{0} \frac{\partial \Phi_{2n}}{\partial \eta} \right] = 0$$

$$\sum_{n=0}^{\infty} \kappa^{2n+1} \left[c_{13}^{0} \frac{\partial U_{2n+1}}{\partial \xi} + c_{13}^{0} \frac{\partial V_{2n+1}}{\partial \eta} + (2n+2) c_{33}^{0} W_{2n+2} + (2n+2) f^{0} \Phi_{2n+2} \right] = 0$$

$$\sum_{n=0}^{\infty} \kappa^{2n+1} \left[e_{31}^{0} \frac{\partial U_{2n+1}}{\partial \xi} + e_{31}^{0} \frac{\partial V_{2n+1}}{\partial \eta} + (2n+2) W_{2n+2} - (2n+2) \varepsilon_{c} f^{0} \Phi_{2n+2} \right] = 0$$

Our new treatment A – 11

■ Generalized plane-stress problem

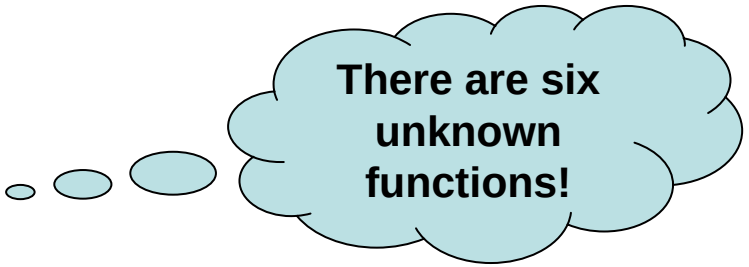
The meaningful lowest-order approximate theory: discarding the second or higher orders of κ in the boundary conditions

$$2U_2 + \frac{\partial W_1}{\partial \xi} + e_{15}^0 f^0 \frac{\partial \Phi_1}{\partial \xi} = 0$$

$$2V_2 + \frac{\partial W_1}{\partial \eta} + e_{15}^0 f^0 \frac{\partial \Phi_1}{\partial \eta} = 0$$

$$c_{13}^0 \frac{\partial U_0}{\partial \xi} + c_{13}^0 \frac{\partial V_0}{\partial \eta} + c_{33}^0 W_1 + f^0 \Phi_1 = 0$$

$$e_{31}^0 \frac{\partial U_0}{\partial \xi} + e_{31}^0 \frac{\partial V_0}{\partial \eta} + W_1 - g^0 \Phi_1 = 0$$



Our new treatment A – 12

■ Generalized plane-stress problem (cont.)

For n = 0, the recursive relations give:

$$U_2 = \frac{1}{2}\left(\frac{\partial^2 U_0}{\partial \tau^2} - c_{11}^0 \frac{\partial^2 U_0}{\partial \xi^2} - c_{66}^0 \frac{\partial^2 U_0}{\partial \eta^2} - c_{126}^0 \frac{\partial^2 V_0}{\partial \xi \partial \eta} - c_{134}^0 \frac{\partial W_1}{\partial \xi} - e_{151}^0 f^0 \frac{\partial \Phi_1}{\partial \xi} \right)$$

$$V_2 = \frac{1}{2}\left(\frac{\partial^2 V_0}{\partial \tau^2} - c_{126}^0 \frac{\partial^2 U_0}{\partial \xi \partial \eta} - c_{66}^0 \frac{\partial^2 V_0}{\partial \xi^2} - c_{11}^0 \frac{\partial^2 V_0}{\partial \eta^2} - c_{134}^0 \frac{\partial W_1}{\partial \eta} - e_{151}^0 f^0 \frac{\partial \Phi_1}{\partial \eta} \right)$$

These two equations supplement the equations derived from the boundary conditions!

Our new treatment A – 13

■ Generalized plane-stress problem (cont.)

Reduce to two equations:

$$\frac{\partial^2 U_0}{\partial \tau^2} - \beta_1 \frac{\partial^2 U_0}{\partial \xi^2} - c_{66}^0 \frac{\partial^2 U_0}{\partial \eta^2} - \beta_2 \frac{\partial^2 V_0}{\partial \xi \partial \eta} = 0$$

$$\frac{\partial^2 V_0}{\partial \tau^2} - \beta_2 \frac{\partial^2 U_0}{\partial \xi \partial \eta} - c_{66}^0 \frac{\partial^2 V_0}{\partial \xi^2} - \beta_1 \frac{\partial^2 V_0}{\partial \eta^2} = 0$$

$$\beta_1 = c_{11}^0 + c_{13}^0 \gamma_1 + e_{31}^0 f^0 \gamma_2$$

$$\beta_2 = c_{126}^0 + c_{13}^0 \gamma_1 + e_{31}^0 f^0 \gamma_2$$

$$\gamma_1 = -(c_{13}\varepsilon_{33} + e_{31}e_{33})/\alpha$$

$$\gamma_2 = (c_{33}e_{31} - c_{13}e_{33})d/(\alpha f)$$

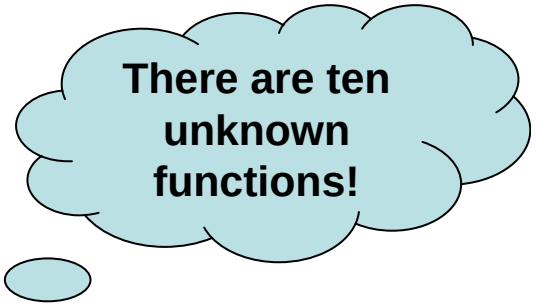
$$\alpha = c_{33}\varepsilon_{33} + e_{33}^2$$

These are identical to that for the generalized plane-stress problem obtained using the conventional stress-relax method.

Our new treatment A – 14

■ Bending problem

The meaningful lowest-order approximate theory: discarding the fourth or higher orders of κ in the boundary conditions



$$U_1+\frac{\partial W_0}{\partial \xi}+e_{15}^0 f^0\frac{\partial \Phi_0}{\partial \xi}+\left(3U_3+\frac{\partial W_2}{\partial \xi}+e_{15}^0 f^0\frac{\partial \Phi_2}{\partial \xi}\right)\kappa^2=0$$

$$V_1+\frac{\partial W_0}{\partial \eta}+e_{15}^0 f^0\frac{\partial \Phi_0}{\partial \eta}+\left(3V_3+\frac{\partial W_2}{\partial \eta}+e_{15}^0 f^0\frac{\partial \Phi_2}{\partial \eta}\right)\kappa^2=0$$

$$c_{13}^0\frac{\partial U_1}{\partial \xi}+c_{13}^0\frac{\partial V_1}{\partial \eta}+2c_{33}^0 W_2+2f^0\Phi_2+\left(c_{13}^0\frac{\partial U_3}{\partial \xi}+c_{13}^0\frac{\partial V_3}{\partial \eta}+4c_{33}^0 W_4+4f^0\Phi_4\right)\kappa^2=0$$

$$e_{31}^0\frac{\partial U_1}{\partial \xi}+e_{31}^0\frac{\partial V_1}{\partial \eta}+2W_2-2g^0\Phi_2+\left(e_{31}^0\frac{\partial U_3}{\partial \xi}+e_{31}^0\frac{\partial V_3}{\partial \eta}+4W_4-4g^0\Phi_4\right)\kappa^2=0$$

Our new treatment A – 15

■ Bending problem (cont.)

For n = 0, the recursive relations give:

$$W_2 = \frac{1}{2}\left[\gamma_{11}\frac{\partial^2 W_0}{\partial \tau^2} + \gamma_{12}\left(\frac{\partial U_1}{\partial \xi} + \frac{\partial V_1}{\partial \eta}\right) + \gamma_{13}\Lambda^0 W_0 + \gamma_{14}\Lambda^0 \Phi_0\right]$$

$$\Phi_2 = \frac{1}{2}\left[\gamma_{21}\frac{\partial^2 W_0}{\partial \tau^2} + \gamma_{22}\left(\frac{\partial U_1}{\partial \xi} + \frac{\partial V_1}{\partial \eta}\right) + \gamma_{23}\Lambda^0 W_0 + \gamma_{24}\Lambda^0 \Phi_0\right]$$

For n = 1, the recursive relations give:

$$U_3 = \frac{1}{6}\left(\frac{\partial^2 U_1}{\partial \tau^2} - c_{11}^0\frac{\partial^2 U_1}{\partial \xi^2} - c_{66}^0\frac{\partial^2 U_1}{\partial \eta^2} - c_{126}^0\frac{\partial^2 V_1}{\partial \xi \partial \eta} - 2c_{134}^0\frac{\partial W_2}{\partial \xi} - 2e_{151}^0 f^0\frac{\partial \Phi_2}{\partial \xi}\right)$$

$$V_3 = \frac{1}{6}\left(\frac{\partial^2 V_1}{\partial \tau^2} - c_{126}^0\frac{\partial^2 U_1}{\partial \xi \partial \eta} - c_{66}^0\frac{\partial^2 V_1}{\partial \xi^2} - c_{11}^0\frac{\partial^2 V_1}{\partial \eta^2} - 2c_{134}^0\frac{\partial W_2}{\partial \eta} - 2e_{151}^0 f^0\frac{\partial \Phi_2}{\partial \eta}\right)$$

Our new treatment A – 16

Bending problem (cont.)

For n = 2, the recursive relations give:

$$W_4 = \frac{1}{12}\left[\gamma_{11}\frac{\partial^2 W_2}{\partial \tau^2} + 3\gamma_{12}\left(\frac{\partial U_3}{\partial \xi} + \frac{\partial V_3}{\partial \eta}\right) + \gamma_{13}\Lambda^0 W_2 + \gamma_{14}\Lambda^0 \Phi_2\right]$$

$$\Phi_4 = \frac{1}{12}\left[\gamma_{21}\frac{\partial^2 W_2}{\partial \tau^2} + 3\gamma_{22}\left(\frac{\partial U_3}{\partial \xi} + \frac{\partial V_3}{\partial \eta}\right) + \gamma_{23}\Lambda^0 W_2 + \gamma_{24}\Lambda^0 \Phi_2\right]$$

Thee six equations rightly supplement the four equations derived from the boundary conditions!

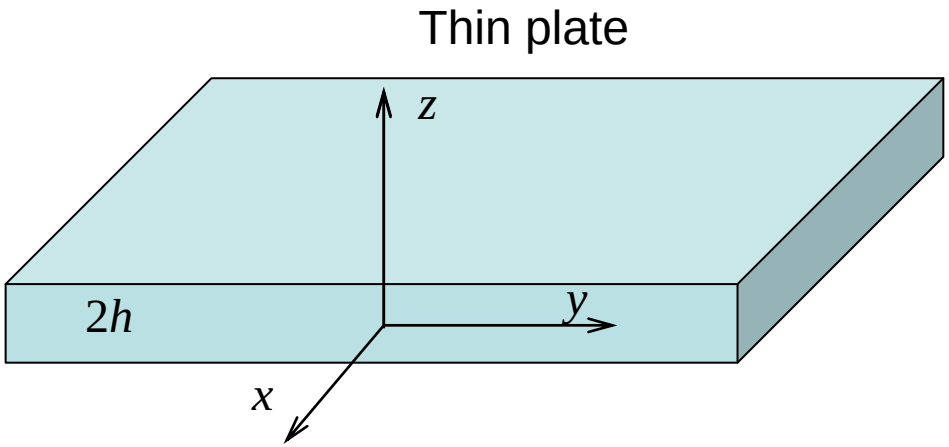
Our new treatment B – 1

State space formulation

State variables/vector:

$$\mathbf{U} = [u_x, u_y, u_z, \phi]^{\mathrm{T}}$$

$$\boldsymbol{\sigma}_z = [\sigma_{xz}, \sigma_{yz}, \sigma_{zz}, D_z]^{\mathrm{T}}$$



State equation:

$$\frac{\partial}{\partial z}\begin{Bmatrix}\mathbf{U}\\ \boldsymbol{\sigma}_z\end{Bmatrix} = \begin{bmatrix}\mathbf{D}_{11} & \mathbf{C}_{33}^{-1}\\ \mathbf{D}_{31} & -\mathbf{D}_{11}^{\mathrm{T}}\end{bmatrix}\begin{Bmatrix}\mathbf{U}\\ \boldsymbol{\sigma}_z\end{Bmatrix} + \frac{\partial^2}{\partial t^2}\begin{Bmatrix}\mathbf{0}\\ \mathbf{K}_\rho\mathbf{U}\end{Bmatrix} = \mathbf{A}\begin{Bmatrix}\mathbf{U}\\ \boldsymbol{\sigma}_z\end{Bmatrix}$$

Sub-matrices D_{ij} contain differential operators with respect to x and y !!!

Tarn JQ (2002) A state space formalism for piezothermoelasticity. International Journal of Solids and Structures, **39**: 5173-5184.

Our new treatment B – 2

- **Formal solution (in operational sense)**

$$\begin{Bmatrix} \mathbf{U}(z) \\ \boldsymbol{\sigma}_z(z) \end{Bmatrix} = \exp(\mathbf{A}z) \begin{Bmatrix} \mathbf{U}(0) \\ \boldsymbol{\sigma}_z(0) \end{Bmatrix}$$

$$\begin{Bmatrix} \mathbf{U}(h) \\ \boldsymbol{\sigma}_z(h) \end{Bmatrix} = \exp(\mathbf{A}h) \begin{Bmatrix} \mathbf{U}(0) \\ \boldsymbol{\sigma}_z(0) \end{Bmatrix} \qquad \begin{Bmatrix} \mathbf{U}(-h) \\ \boldsymbol{\sigma}_z(-h) \end{Bmatrix} = \exp(-\mathbf{A}h) \begin{Bmatrix} \mathbf{U}(0) \\ \boldsymbol{\sigma}_z(0) \end{Bmatrix}$$

Our new treatment B – 3

- **Power series expression**

$$\begin{Bmatrix} \mathbf{U}(h) \\ \boldsymbol{\sigma}_z(h) \end{Bmatrix} = \left[\mathbf{I} + \mathbf{A}h + \frac{1}{2}\mathbf{A}^2h^2 + \cdots + \frac{1}{n!}\mathbf{A}^n h^n + \cdots \right] \begin{Bmatrix} \mathbf{U}(0) \\ \boldsymbol{\sigma}_z(0) \end{Bmatrix}$$

$$\begin{Bmatrix} \mathbf{U}(-h) \\ \boldsymbol{\sigma}_z(-h) \end{Bmatrix} = \left[\mathbf{I} - \mathbf{A}h + \frac{1}{2}\mathbf{A}^2h^2 + \cdots + \frac{(-1)^n}{n!}\mathbf{A}^n h^n + \cdots \right] \begin{Bmatrix} \mathbf{U}(0) \\ \boldsymbol{\sigma}_z(0) \end{Bmatrix}$$

- **The nth-order plate theory**

$\boldsymbol{\sigma}_z(\pm h) = 0$

$$\boldsymbol{\sigma}_z(0) + [\mathbf{A}_{21}^1\mathbf{u}(0) + \mathbf{A}_{22}^1\boldsymbol{\sigma}_z(0)]h + \frac{1}{2}[\mathbf{A}_{21}^2\mathbf{u}(0) + \mathbf{A}_{22}^2\boldsymbol{\sigma}_z(0)]h^2$$

$$+ \cdots + \frac{1}{n!}[\mathbf{A}_{21}^n\mathbf{u}(0) + \mathbf{A}_{22}^n\boldsymbol{\sigma}_z(0)]h^n = \mathbf{0},$$

$$\boldsymbol{\sigma}_z(0) - [\mathbf{A}_{21}^1\mathbf{u}(0) + \mathbf{A}_{22}^1\boldsymbol{\sigma}_z(0)]h + \frac{1}{2}[\mathbf{A}_{21}^2\mathbf{u}(0) + \mathbf{A}_{22}^2\boldsymbol{\sigma}_z(0)]h^2$$

$$+ \cdots + \frac{(-1)^n}{n!}[\mathbf{A}_{21}^n\mathbf{u}(0) + \mathbf{A}_{22}^n\boldsymbol{\sigma}_z(0)]h^n = \mathbf{0},$$

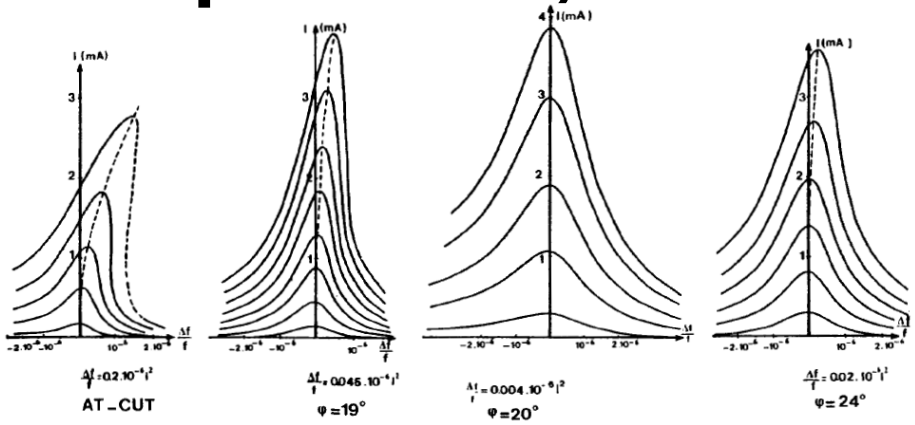
Six equations, six unknown functions!

Outline

- Theory of piezoelectricity: A brief review
- Mindlin's theory for piezoelectric plates
- Different treatments for linear plate analysis
- **Nonlinear plate analysis**
- Concluding remarks

Nonlinear Analysis is Import.

- **Harmonic generation, amplitude-frequency effect, intermodulation, …**
- **Sensitivities to external or internal perturbations (temperature, force, pressure, acceleration, electric field, …)**

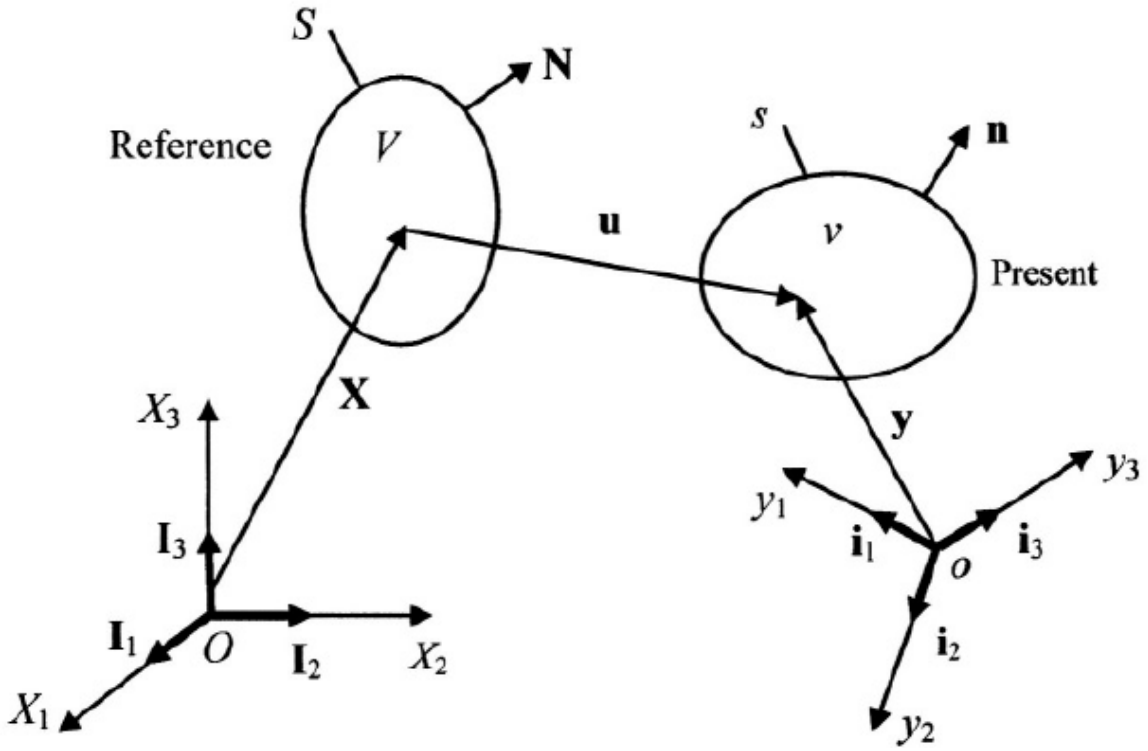


Amplitude-frequency effect of doubly-rotated cut (5MH, 3rd overtone)

Gagnepain JJ (1981) Nonlinear properties of quartz crystal and quartz resonators: A review. Proceedings of the 35th Ann. Freq. Control Symposium, pp.14-30.

Nonlinear Theory – 1

- **Referential and present configurations**



Two different coordinates

Nonlinear Theory – 2

Basic formulations

Equations of motion and charge:

$$K_{Lk,L} + \rho_0 f_k = \rho_0 \ddot{y}_k$$

$$\mathcal{D}_{K,K} = 0$$

Constitutive relations:

$$K_{Lk} = y_{k,K}\rho_0 \frac{\partial \psi}{\partial S_{KL}} + JX_{L,j}\varepsilon_0 \left(E_j E_k - \frac{1}{2} E_i E_i \delta_{jk} \right)$$

$$\mathcal{D}_K = \varepsilon_0 J X_{K,k} E_k - \rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}$$

$$J = \det(y_{k,K})$$

$$u_K = y_K - X_K \quad (y_K = y_j \delta_{jM})$$

Gradient relations:

$$S_{KL} = (y_{k,K} y_{k,L} - \delta_{KL})/2$$

$$E_i = -\phi_{,i}, \; \mathcal{E}_K = -\phi_{,K}$$

Boundary conditions:

$$N_L K_{Lj} = t_j \;\text{ or }\; u_K = \bar{u}_K$$

$$N_K \mathcal{D}_K = \sigma \;\text{ or }\; \phi = \bar{\phi}$$

Nonlinear Theory – 3

- **Energy density function**

$$\rho_0\psi = \frac{1}{2}c_{ABCD}S_{AB}S_{CD} - e_{ABC}\mathcal{E}_A S_{BC} - \frac{1}{2}\chi_{AB}\mathcal{E}_A\mathcal{E}_B + \frac{1}{6}c_{ABCDEF}S_{AB}S_{CD}S_{EF}$$

For case of large deformations and weak electric fields

- **Stress in terms of displacement gradient and electric field**

$$K_{LM} = c_{LMRS}u_{R,S} - e_{KLM}\mathcal{E}_K + c^e_{LMRSKN}u_{R,S}u_{K,N}$$

$$\mathcal{D}_K = e_{KRS}u_{R,S} + \varepsilon_{KL}\mathcal{E}_L$$

$$\varepsilon_{KL} = \varepsilon_0\delta_{KL} + \chi_{KL}$$

$$c^e_{LMRSKN} = \frac{1}{2}(c_{LMRSKN} + c_{LMNS}\delta_{KR} + c_{LNRS}\delta_{KM})$$

Dorfmann A, Ogden RW (2006) Nonlinear electroelastic deformations. Journal of Elasticity, **82**(2): 99-127.

Nonlinear Structural Theories

Tiersten HF (1976) Analysis of nonlinear resonance in thickness-shear and trapped-energy resonators. Journal of the Acoustical Society of America, **59**: 866-878.

Thompson RB, Tiersten HF (1977) Harmonic generation of longitudinal elastic waves. Journal of the Acoustical Society of America, **62**: 33-37.

Tiersten HF (1983) Nonlinear extensional vibrations of quartz rods. Journal of the Acoustical Society of America, **73**: 2022-2033.

Yang JS, et al (2003) Two-dimensional equations for electroelastic plates with relatively large shear deformations. IEEE Transactions on Ultrasonics, Ferroelectricity, and Frequency Control, **50**: 765-772.

Yang JS, Shen XC (2008) Coupling to extension in a thickness-shear resonator due to relatively large thickness-shear deformation. IEEE Transactions on Ultrasonics, Ferroelectricity, and Frequency Control, **55**: 726-729.

Yang JS (2008) Two-dimensional equations for electroelastic plates with relatively large in-plane shear deformation and nonlinear mode coupling in resonant piezoelectric devices. Acta Mechanica, **196**: 103-111.

Our primary idea

- **Series expansions!**

  Important observations:

  1) The governing equations still give recursive relations, although they now become nonlinear!

  2) The boundary conditions have the same structures as the linear problem, although they are also nonlinear!

A closed set of nonlinear PDEs can be obtained for different orders of approximation!

Outline

- Theory of piezoelectricity: A brief review
- Mindlin's theory for piezoelectric plates
- Different treatments for linear plate analysis
- Nonlinear plate analysis
- **Concluding remarks**

Summary and Remarks

- Correctly modeling plate structures is of great significance in practice.
- There are many different theories available in literature, most starting from certain assumption on the distributions of field quantities along the thickness, usually in polynomial forms.
- Systematic series expansion of field quantities enables a rational derivation of plate equations, which however requires appropriate truncation of the series.
- We have shown several different treatments, some are rather primary, and comprehensive comparison and scrutinizing is still very much expected.

Questions

- How is Mindlin's plate theory related to Reissner's plate theory?
- What differences exist between Mindlin's treatment and BJO treatment?
- Can the methods be applied to functionally graded material plates? How?
- How are the methods extended to shell structures?
- ….

A few related papers of ours

Zhang CL, Yang JS, Chen WQ (2009) Theoretical modeling of frequency-dependent magnetoelectric effects in laminated multiferroic plates. IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, **56**(12): 2750-2759.

Chen WQ (2011) Surface effect on Bleustein-Gulyaev wave in a piezoelectric half-space. Theoretical and Applied Mechanics Letters, **1**(4): no. 041001(4pp).

Zhu J, Chen WQ (2011) Thickness-twist and face-shear waves in piezoelectric plates of monoclinic crystals. IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control, **58**(12): 2763-2767.

Zhu J, Chen WQ, Ye GR (2012) Reverberation-ray matrix analysis for wave propagation in multiferroic plates with imperfect interfacial bonding. Ultrasonics, **52**(1): 125-132.

Zhou YY, Lü CF, Chen WQ (2012) Bulk wave propagation in layered piezomagnetic/piezoelectric plates with initial stresses or interface imperfections. Composite Structures, **94**(9): 2736-2745.

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